Spectrahedra and Their Shadows

Der Fakultät für Mathematik und Informatik
der Universität Leipzig
eingereichte
Habilitationsschrift
zur Erlangung des akademischen Grades

doctor rerum naturalium habilitatus
(dr. rer. nat. habil.)

vorgelegt von

Dr. rer. nat. Tim Netzer

geboren am 16.3.1980 in Leutkirch

Leipzig, den 28. Oktober 2011

Selbständigkeitserklärung

Hiermit erkläre ich, die vorliegende Habilitationsschrift selbständig und ohne unzulässige fremde Hilfe angefertigt zu haben. Ich habe keine anderen als die angeführten Quellen und Hilfsmittel benutzt und sämtliche Textstellen, die wörtlich oder sinngemäß aus veröffentlichten oder unveröffentlichten Schriften entnommen wurden, und alle Angaben, die auf mündlichen Auskünften beruhen, als solche kenntlich gemacht. Ebenfalls sind alle von anderen Personen bereitgestellten Materialien oder erbrachten Dienstleistungen als solche gekennzeichnet.

Leipzig, den 28. Oktober 2011

for Miranda

Contents

Introduction **7**

I Spectrahedra **11**

1 Definitions and Preliminaries **13**
- 1.1 Definitions 13
- 1.2 Motivation 15
- 1.3 First Properties and Constructions 17

2 Real Zero Polynomials & Rigid Convexity **21**
- 2.1 Definitions 21
- 2.2 The Multivariate Hermite Matrix 25
- 2.3 Hyperbolic Polynomials 29
- 2.4 Renegar Derivatives 32
- 2.5 Smooth Approximation 34

3 Determinantal Representations **37**
- 3.1 Preliminaries 37
- 3.2 Bounds on the Size 38
  - 3.2.1 Upper Bounds 39
  - 3.2.2 Lower Bounds 42
  - 3.2.3 Representations of Hyperbolic Polynomials 45
- 3.3 Polynomials without Representations 46
  - 3.3.1 Almost no Polynomial has a Representation 46
  - 3.3.2 Convex Cones Examples 48
  - 3.3.3 Compact Examples 50
  - 3.3.4 Renegar Derivatives 51

3.3.5 Polynomials of which no Power has a Representation . 52
3.4 Polynomials with Representations 57
3.4.1 A General Construction Method 57
3.4.2 Representing Powers of Real Zero Polynomials 63
3.4.3 Rational Representations 72

II Spectrahedral Shadows 77

4 Definitions and First Results 79
4.1 Definitions 79
4.2 Lasserre Relaxations 84
4.3 Convex Hulls 91
4.4 The Results of Helton and Nie 98

5 Limitations of the Relaxation Methods 101
5.1 Preliminaries 101
5.2 Main Result 104
5.3 Examples 108

6 Closures and Interiors 111
6.1 The Closure 111
6.2 Some Helpful Results 112
6.3 Interiors 113

References 121

Deutsche Zusammenfassung 127

Introduction

The theory of spectrahedra and their shadows is a fascinating and active area of research. The interest in it was triggered by the development of semidefinite programming towards the end of the last century.

Semidefinite programming is a generalization of linear programming. In linear programming, one tries to optimize a linear function under linear constraints. So the set over which the function is optimized is a polyhedron. A lot of optimization problems from the real world can be transformed into linear programming problems. So it is not surprising that linear programming has been extensively studied since at least the second world war. There exist efficient algorithms to solve such problems.

In semidefinite programming, the linear constraints are replaced by the condition that a linear combination of some matrices is positive semidefinite. The feasible sets that occur in this way are still convex, but not necessarily polyhedral. They are called *spectrahedra*. Allowing for those more general constraints of course broadens the area of application. Luckily, there are still efficient algorithms to solve these more general optimization problems. This is what makes semidefinite programming so interesting.

The development of semidefinite programming of course asks for a thorough investigation of the underlying theoretical framework. One of the most interesting open questions concerns the characterization of spectrahedra. It might already be complicated to decide whether a given set is a polyhedron in certain cases, but for spectrahedra it is much worse. Even for a two- or three-dimensional explicitly given set it might be unclear whether it is a spectrahedron or not.

There is groundbreaking work of Helton and Vinnikov on this question. They introduce the notion of a *rigidly convex set*, and show that spectrahedra are rigidly convex. Whether a set is rigidly convex is easier to decide in general. The question whether a rigidly convex set is a spectrahedron is then linked to the problem of writing certain polynomials as determinants of linear matrix polynomials. This establishes a translation of the original geometric question into an algebraic problem. In the two-dimensional case, Helton and

Vinnikov solve this problem to the positive, and the classification of two-dimensional spectrahedra can thus be considered as completed: spectrahedra and rigidly convex sets are the same in the plane. In higher dimensions, the classification of spectrahedra is still open. It can however be shown that the strong algebraic result of Helton and Vinnikov does not carry over directly.

Also of interest are projections of spectrahedra, called *spectrahedral shadows*. In contrast to polyhedra, such projections are not necessarily of the same type again, i.e. they are not necessarily spectrahedra. Still they are feasible for semidefinite programming. One simply has to optimize over the original spectrahedron, which might involve some additional variables. The only known necessary conditions for a set to be a spectrahedral shadow is being convex and semi-algebraic. Whether these two conditions are also sufficient is an open problem.

In this work we want to examine the above-mentioned questions in more detail. In the first part we consider spectrahedra. We introduce the whole framework and explain the most important results. The main focus will then be on the algebraic part, i.e. on the problem of writing certain polynomials as determinants of linear matrix polynomials. We show that this is most often not possible in dimension three or higher. We explicitly construct examples of such polynomials without determinantal representations. There exist surprisingly simple examples.

We then examine whether some power of a polynomial admits a determinantal representation. This is linked to the question of representing a multivariate Hermite matrix as a sum of squares of matrices: if some power of the polynomial has a determinantal representation, then its Hermite matrix is a sum of squares. This condition can for example be checked numerically quite well, and thus yields new counterexamples.

Conversely, we try to construct a determinantal representation for a polynomial, or some multiple, from a sums of squares decomposition of its Hermite matrix. The method involves an extension of a graded morphism from a submodule of a free module to the whole module. Checking whether this can be done amounts only to solving a system of linear equations. Note that there are very few methods to explicitly construct determinantal representations so far, in particular in dimension three or higher.

We then characterize polynomials of which some power has a determinantal representation, in terms of a non-commutative algebra having a finite dimensional representation. We deduce that the desired representations exist for quadratic polynomials. The representations thereby emerge explicitly, and we describe them up to unitary equivalence.

We finally show that each of the considered polynomials admits a *rational* determinantal representation.

In the second part of this work we examine spectrahedral shadows. We again first give a general introduction, including a unified account of the most important results and concepts. We then focus on the so-called Lasserre relaxation method. This method allows to construct a sequence of spectrahedral shadows that approximate a given set. An interesting question is whether this approximation is exact. We show how a geometric property of the original set prohibits this exactness: if the set has a non-exposed face, then no Lasserre relaxation can be exact. The result can also be formulated in a purely real-algebraic setup, talking only about sums of squares representations of certain nonnegative polynomials.

We next deal with non-closed sets. We show how many such non-closed sets can be realized as spectrahedral shadows: One can remove suitably parametrized faces from a spectrahedral shadow and obtain a spectrahedral shadow again. In particular, the interior of a spectrahedral shadow is a spectrahedral shadow. Also the closure of a spectrahedral shadow turns out to be a spectrahedral shadow again.

The results from this work are mostly already published. They appear in Gouveia and Netzer [14], Netzer [37], Netzer, Plaumann and Schweighofer [38], Netzer, Plaumann and Thom [39], Netzer and Sinn [40] and Netzer and Thom [41]. I would like to thank all my coauthors for the pleasant collaboration. Further I want to thank Alexander Prestel, Claus Scheiderer and Konrad Schmüdgen for their constant support, that helped me a lot.

Part I

Spectrahedra

Chapter 1

Definitions and Preliminaries

1.1 Definitions

A *spectrahedron* is a set defined by a linear matrix inequality. Formally, let $M_0, M_1, \dots, M_n \in \operatorname{Her}_k(\mathbb{C})$ be complex hermitian matrices of size k. The expression

$$\mathcal{M} := M_0 + x_1 M_1 + \cdots + x_n M_n$$

is called a *(hermitian) linear matrix polynomial.* Here, $\underline{x} = (x_1, \dots, x_n)$ is a tuple of commuting variables. In the special case that all matrices $M_i \in \operatorname{Sym}_k(\mathbb{R})$ are real symmetric matrices, we call the matrix polynomial a *symmetric linear matrix polynomial.* We will in the following always assume that matrix polynomials are hermitian, and state explicitly if we restrict ourselves to symmetric ones. The *size* of a linear matrix polynomial is the size k of its coefficient matrices. The number n of variables is called the *dimension* of the matrix polynomial.

For a linear matrix polynomial $\mathcal{M}$ of dimension n we consider the set of points from $\mathbb{R}^n$ where $\mathcal{M}$ is positive semidefinite:

$$\mathcal{S}(\mathcal{M}) := \{a \in \mathbb{R}^n \mid \mathcal{M}(a) = M_0 + a_1 M_1 + \cdots + a_n M_n \succeq 0\}.$$

Such a set is called a *spectrahedron.* The name is justified in Ramana and Goldmann [48] as follows: $\mathcal{S}(\mathcal{M})$ is defined via the spectrum of $\mathcal{M}$ (i.e. it consists of all points at which $\mathcal{M}$ has nonnegative spectrum), and the notion generalizes that of a polyhedron. In fact if $\ell_1, \dots, \ell_k \in \mathbb{R}[\underline{x}]_1$ are linear polynomials, consider

$$\mathcal{M} = \begin{pmatrix} \ell_1 & & \\ & \ddots & \\ & & \ell_k \end{pmatrix}$$

which is a symmetric linear matrix polynomial (with all matrices M_i diagonal). It is clear that

$$\mathcal{S}(\mathcal{M}) = \{a \in \mathbb{R}^n \mid \ell_1(a) \geq 0, \dots, \ell_k(a) \geq 0\}.$$

So each polyhedron is a spectrahedron. It turns out that spectrahedra share some interesting properties with polyhedra, but on the other hand enlarge that class greatly. Whereas polyhedra are well examined objects, the theory of spectrahedra is not widely developed. For example, there is still no satisfying procedure to check whether a set is a spectrahedron, apart from dimension two.

Note that one can also define spectrahedra to be intersections of the cone of positive semidefinite matrices with an affine linear subspace. This is clear when considering $M_1, \dots, M_n$ as a spanning system of a subspace and M_0 as an affine translation vector. In the same way polyhedra can be seen as affine linear intersections of the positive orthant in some $\mathbb{R}^k$.

Here is a first explicit example.

Example 1.1.1. Let $n = 2$ and consider

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, M_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$\mathcal{M} = M_0 + x_1 M_1 + x_2 M_2 = \begin{pmatrix} 1 + x_1 & x_2 \\ x_2 & 1 - x_1 \end{pmatrix},$$

and one checks that

$$\mathcal{S}(\mathcal{M}) = \left\{(a_1, a_2) \in \mathbb{R}^2 \mid a_1^2 + a_2^2 \leq 1\right\}$$

is the unit disk.

1.2 Motivation

Spectrahedra form a most interesting class of convex sets, somewhere between "easy" and "very complicated". The class of spectrahedra contains the well-studied and important class of polyhedra. It contains however many more convex sets, as we will see in the further process. Still, spectrahedra are well manageable due to their explicit description via matrix polynomials.

One of the main interests in spectrahedra stems from optimization theory. The problem to find the minimum of an affine linear function on a polyhedron (and possibly the points where this minimum is attained) is known as *linear programming*. Linear programming turns out to be extremely useful for many kinds of problems. Furthermore, there exists a well developed theory of linear programming, including efficient algorithms (such as interior point methods and the simplex algorithm) and a good duality theory. The existing literature on linear programming is extensive; for a detailed survey see for example Murty [34].

Replacing polyhedra by spectrahedra in the optimization problem clearly broadens the area of application. Many interesting problems can indeed by understood as optimization problems over spectrahedra, but not over polyhedra. Examples come from polynomial optimization, combinatorial optimization, non-convex optimization and control theory. Optimization over spectrahedra is called *semidefinite programming*. So semidefinite programming is a generalization of linear programming. Luckily, there exists a good duality theory and efficient algorithms for semidefinite programming as well. For more details on semidefinite programming see for example Ben-Tal and Nemirovski [6], Nemirovski [35], Nesterov and Nemirovski [36], Todd [58], Vandenberghe and Boyd [59] and Wolkowicz, Saigal and Vandenberghe [61].

As already mentioned, there is still no complete characterization of spectrahedra. From the practical viewpoint of optimization, an easy characterization is however most desirable. Also algorithmic problems, like how to find a representing matrix polynomial, are of great interest, and have not been addressed extensively. So the research on spectrahedra is not only attractive from a theoretical point of view, but also well motivated in terms of applications.

We finish this section with an interesting application of semidefinite programming. The approach is due to Lovász [30], and our explanation is taken from Barvinok [4], Section IV.11.

Example 1.2.1. Let $G = (V, E)$ be an undirected graph on k vertices $V = \{1, \ldots, k\}$, containing no loops or multiple edges. A *clique* of G is a subset $W \subseteq V$ such that any two distinct elements from W are connected by an edge. The *clique number* $\omega(G)$ of G is the largest cardinality of a clique.

A *coloring* of G is an assignment of a color to each vertex of G, such that two vertices that are connected by an edge never have the same color. The *chromatic number* $\chi(G)$ of G is the smallest number of colors needed for a coloring. It is easy to see that

$$\omega(G) \leq \chi(G)$$

holds for all graphs. It is known that finding $\omega(G)$ and $\chi(G)$ is computationally hard. What *can* be done with a semidefinite program however, is computing a number $\vartheta(G)$ that always lies between $\omega(G)$ and $\chi(G)$:

$$\omega(G) \leq \vartheta(G) \leq \chi(G).$$

This number $\vartheta(G)$ is called *Lovász's theta number* of G, and can be defined as the optimal value of the following optimization problem:

$$\begin{aligned} & \vartheta(G) := \sup \sum_{i,j} x_{ij} \\ \text{where} \quad & (x_{ij}) \in \mathrm{Sym}_k(\mathbb{R}) \\ & (x_{ij}) \succeq 0 \\ & x_{ij} = 0 \quad \text{whenever } (i,j) \notin E \\ & \sum_i x_{ii} = 1. \end{aligned}$$

The feasible set of this optimization problem is an affine linear section of the cone of positive semidefinite matrices; it can thus be understood as a spectrahedron as we defined it. The objective function is linear. Thus $\vartheta(G)$ is indeed the optimal value of a semidefinite programming problem. The fact that $\vartheta(G)$ lies between $\omega(G)$ and $\chi(G)$ is not too hard to prove. We refer the reader to Barvinok's book [4].

1.3 First Properties and Constructions

We collect some basic properties of spectrahedra. Note that Properties 1.-3. are standard properties of polyhedra.

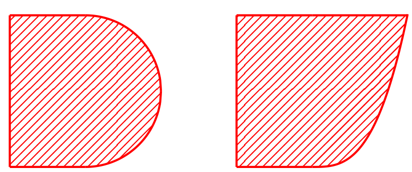
1. From the definition it is easily seen that spectrahedra are convex, closed and semi-algebraic sets.

2. Spectrahedra are even *basic closed semi-algebraic*, i.e. definable by finitely many simultaneous polynomial inequalities. Indeed if $\mathcal{M}$ is a linear matrix polynomial and $p_1, \ldots, p_m \in \mathbb{R}[\underline{x}]$ are its principal minors, then

$$\mathcal{S}(\mathcal{M}) = \{a \in \mathbb{R}^n \mid p_1(a) \geq 0, \ldots, p_m(a) \geq 0\}.$$

   The picture on the left in Figure 1.1 shows the union of the unit disk and a square. This is a typical example of a semi-algebraic set that is *not* basic closed. So it is not a spectrahedron.

Figure 1.1:



3. Ramana and Goldmann [48] show that spectrahedra have only *exposed faces*.

   A *face* of a convex set $S \subseteq \mathbb{R}^n$ is a nonempty convex subset $F \subseteq S$, such that for $x, y \in S$ and $\lambda \in [0,1]$, $\lambda x + (1-\lambda)y \in F$ implies $x, y \in F$. So the faces different from S are certain extremal subsets of the boundary of S. A special case is that of an extreme point (i.e. if $F = \{a\}$ is a singleton). A face F of S is *exposed*, if there is an affine linear function ℓ on $\mathbb{R}^n$, with $\ell \geq 0$ on S and

$$F = \{a \in S \mid \ell(a) = 0\}.$$

   In case that $F \neq S$ this is the same as saying that there is a supporting hyperplane of S that touches S precisely in F.

Ramana and Goldmann first show that the faces of a spectrahedron can be parametrized via the kernels of the defining matrix polynomial. Indeed if $U \subseteq \mathbb{R}^k$ is a subspace, then

$$F_U := \{a \in \mathcal{S}(\mathcal{M}) \mid U \subseteq \ker \mathcal{M}(a)\}$$

is a face of $\mathcal{S}(\mathcal{M})$, and each face is of that form. They deduce that each face is exposed. Indeed if $u_1, \ldots, u_r$ is a basis of U, then the affine linear function

$$\ell(a) := u_1^t \mathcal{M}(a) u_1 + \cdots + u_r^t \mathcal{M}(a) u_r$$

is nonnegative on $\mathcal{S}(\mathcal{M})$ and exposes F_U.

The picture on the right in Figure 1.1 shows the basic closed semi-algebraic set defined by the inequalities $-1 \leq x_1, 0 \leq x_2 \leq 1, x_1^3 \leq x_2$. The origin is a non-exposed face of that set, and it is thus not a spectrahedron.

4. Each spectrahedron is definable by a symmetric linear matrix polynomial. Indeed let $\mathcal{M}$ be a hermitian linear matrix polynomial, and decompose $\mathcal{M} = \mathcal{R} + i\mathcal{I}$ into a symmetric linear matrix polynomial $\mathcal{R}$ and a real skew-symmetric matrix polynomial $\mathcal{I}$. Then

$$\mathcal{M}' := \left(\begin{array}{c|c} \mathcal{R} & \mathcal{I} \\ \hline -\mathcal{I} & \mathcal{R} \end{array}\right)$$

is a symmetric linear matrix polynomial, and one has

$$\mathcal{S}(\mathcal{M}) = \mathcal{S}(\mathcal{M}').$$

This easy fact is for example explained in Ramana and Goldmann [48], and will also follow from Lemma 3.2.14 below.

5. If a spectrahedron has nonempty interior in $\mathbb{R}^n$, then it is definable by a *strictly feasible* linear matrix polynomial, i.e. a linear matrix polynomial $\mathcal{M}$ with $\mathcal{M}(a) \succ 0$ for some $a \in \mathbb{R}^n$. Here, $\succ$ denotes positive definiteness. Indeed any defining matrix polynomial can be reduced to such a strictly feasible matrix polynomial. This is for example proven in Ramana and Goldmann [48], Corollary 5, or Helton and Vinnikov [21], Lemma 2.3. The argument is as follows:

We can assume without loss of generality that the origin is in the interior of $\mathcal{S}(\mathcal{M})$. Thus $M_0 \pm \varepsilon M_i$ is positive semidefinite, for some $\varepsilon > 0$ and all $i = 1, \ldots, n$. So whenever $v \in \mathbb{R}^k$ is in the kernel of M_0, then

$$0 \leq v^t \left(M_0 \pm \varepsilon M_i\right) v = \pm\varepsilon v^t M_i v.$$

So $v^t M_i v = 0$ and thus also $v^t \left(M_0 \pm \varepsilon M_i\right) v = 0$. Since $M_0 \pm \varepsilon M_i$ is positive semidefinite, this implies $\left(M_0 \pm \varepsilon M_i\right) v = 0$, which finally yields $M_i v = 0$. So the kernel of M_0 is contained in the kernel of each M_i. After a suitable change of coordinates, all matrices split off a block of zeros. When deleting this block, the new matrix M_0 is positive definite, which finishes the argument.

After conjugation with a suitable matrix, we then get

$$M_0 = I,$$

the identity matrix. A linear matrix polynomial with $M_0 = I$ will be called *monic*.

We will restrict ourselves to monic linear matrix polynomials throughout this first part of the work! This means restricting ourselves to spectrahedra that contain the origin in the interior. Since we can always replace the ambient space of a spectrahedron by its affine hull, and make a suitable translation, this is not a real restriction.

The class of spectrahedra is obviously closed under certain elementary constructions. For instance, the intersection of two spectrahedra is again a spectrahedron. If $S = \mathcal{S}(\mathcal{M})$ and $T = \mathcal{S}(\mathcal{N})$, then

$$S \cap T = \mathcal{S}(\mathcal{M} \oplus \mathcal{N}),$$

where

$$\mathcal{M} \oplus \mathcal{N} = \left(\begin{array}{c|c} \mathcal{M} & \\ \hline & \mathcal{N} \end{array}\right)$$

is the diagonal block matrix built of $\mathcal{M}$ and $\mathcal{N}$. Also the inverse image of a spectrahedron under an affine linear map is again a spectrahedron. This is immediately clear from the definition. Thus finally the cartesian product of two spectrahedra is again a spectrahedron. It can in fact be realized as the intersection of two inverse images.

In contrast to polyhedra, the *image* of a spectrahedron under an affine linear map is not necessarily a spectrahedron, as we will see in Part II of this work in detail. Also the convex hull of two spectrahedra is not necessarily again a spectrahedron.

The next Chapter is devoted to explaining one of the most important properties of spectrahedra, the so-called *rigid convexity*.

Chapter 2

Real Zero Polynomials & Rigid Convexity

2.1 Definitions

The notion of *rigid convexity* has been introduced by Helton and Vinnikov in their seminal paper [21]. They show that each spectrahedron is rigidly convex, so the notion works well for excluding sets from being spectrahedra. Conversely, they proof that each rigidly convex set *in the plane* is a spectrahedron. This deep result provides a complete characterization of two-dimensional spectrahedra. Whether the same also works in higher dimensions is one of the most important open problems in the area of convex algebraic geometry.

To explain rigid convexity, we start with the notion of a real zero polynomial:

Definition 2.1.1. A polynomial $p \in \mathbb{R}[\underline{x}]$ is called a *real zero polynomial*, if $p(0) = 1$ and if for all $a \in \mathbb{R}^n$ and $\lambda \in \mathbb{C}$,

$$p(\lambda \cdot a) = 0 \Rightarrow \lambda \in \mathbb{R}.$$

Note that the second condition means that p has only real roots, if restricted to any line through the origin. The geometric way to state this is the following. If

$$\mathcal{V}_{\mathbb{R}}(p) = \{a \in \mathbb{R}^n \mid p(a) = 0\}$$

denotes the real zero set of p, then any generic line through the origin in $\mathbb{R}^n$ intersects $\mathcal{V}_{\mathbb{R}}(p)$ in $\deg(p)$ many points. We will also use the following notation: for $a \in \mathbb{R}^n$ we denote by p_a the univariate polynomial

$$p_a(t) := p(t \cdot a).$$

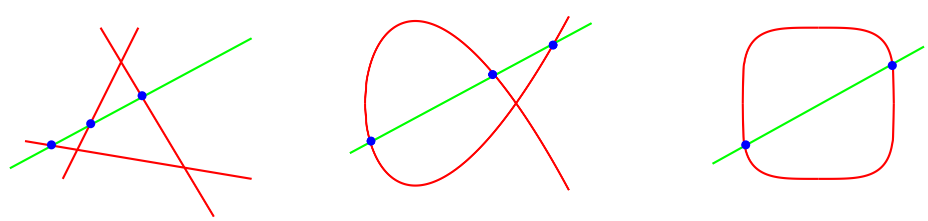
So p is a real zero polynomial if and only if all p_a split into linear factors over $\mathbb{R}$. The following definition will be used later on.

Definition 2.1.2. A real zero polynomial $p \in \mathbb{R}[\underline{x}]$ is called *smooth* if none of the polynomials $p_a(t)$ with $a \in \mathbb{R}^n \setminus \{0\}$ has a multiple root. This includes possible roots at infinity, i.e. we must have $\deg(p_a) \geq \deg(p) - 1$ for all $a \neq 0$.

Remark 2.1.3. It is easily seen that a product pq of polynomials is real zero if and only if both p and q are real zero.

Example 2.1.4. An easy example for a real zero polynomial is a (suitably scaled) product of linear polynomials; a nontrivial example is the polynomial $p = x_1^3 - x_1^2 - x - x_2^2 + 1 \in \mathbb{R}[x_1, x_2]$. An example of a polynomial that is not a real zero polynomial is $q = 1 - x_1^4 - x_2^4$. See Figure 2.1 for the zero set of a product of linear polynomials, the zero set of p, and the zero set of q. The intersection points with a line through the origin are marked as dots.

Figure 2.1:



A rigidly convex set is now just the ellipsoid within the innermost ring of zeros of some real zero polynomial. Here is an exact definition:

Definition 2.1.5. Let $p \in \mathbb{R}[\underline{x}]$ be a real zero polynomial. Then the set

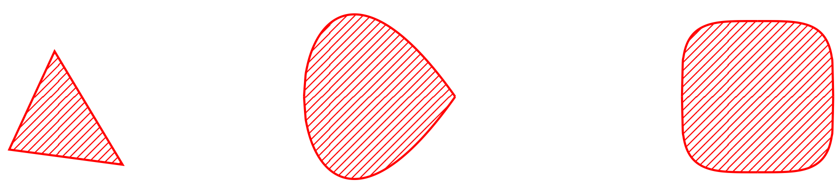
$$\mathcal{R}(p) := \{a \in \mathbb{R}^n \mid p_a \text{ has no roots in the interval } [0, 1)\}$$

is called the *rigidly convex set* defined by p.

Note that the rigidly convex set defined by p consists of all line segments joining the origin and the first zero of p, in each direction.

Example 2.1.6. The set on the left in Figure 2.2 is the rigidly convex set defined by the polynomial $p = x_1^3 - x_1^2 - x - x_2^2 + 1$. The set on the right is $S = \{(a_1, a_2) \in \mathbb{R}^2 \mid a_1^4 + a_2^4 \leq 1\}$. It is easily seen to *not* be rigidly convex. Indeed if it was, there would be a real zero polynomial vanishing on the boundary of S. This polynomial would need to contain $q = 1 - x_1^4 - x_2^4$ as a factor, which is impossible for a real zero polynomial.

Figure 2.2:



Remark 2.1.7. There is a strong connection between real zero polynomials and so-called *hyperbolic polynomials*. We will explain this connection in Section 2.3 in more detail. At this point just note that a result of Gårding [16] on hyperbolic polynomials immediately implies that rigidly convex sets are indeed convex. That result can also be deduced from the important result of Helton and Vinnikov, Theorem 2.1.12 below.

Also note that a result of Renegar [49] implies that rigidly convex sets are basic closed semi-algebraic and have only exposed faces.

As already mentioned, each spectrahedron is a rigidly convex set. This interesting fact was first observed by Helton and Vinnikov [21]. We explain the approach from Netzer and Thom [41]. The following easy lemma will be useful at several points in the following work.

Lemma 2.1.8. *Let $\mathcal{M} = I + x_1 M_1 + \cdots + x_n M_n$ be a linear matrix polynomial and $p := \det \mathcal{M} \in \mathbb{R}[\underline{x}]$ its determinant. Then for each $a \in \mathbb{R}^n$, the nonzero eigenvalues of $a_1 M_1 + \cdots + a_n M_n$ are in one to one correspondence with the zeros of the univariate polynomial $p_a(t) := p(t \cdot a)$, counting multiplicities. The correspondence is given by the rule $\lambda \mapsto -\frac{1}{\lambda}$.*

Proof. Fix $a \in \mathbb{R}^n$ and let c_a denote the characteristic polynomial of the hermitian matrix $a_1M_1 + \cdots + a_nM_n$. For any $\lambda \neq 0$ we have

$$\begin{aligned} c_a(\lambda) &= \det\left(-\lambda I + a_1M_1 + \cdots + a_nM_n\right) \\ &= (-\lambda)^k p\left(\frac{a}{-\lambda}\right) = (-\lambda)^k p_a\left(-\frac{1}{\lambda}\right). \end{aligned}$$

We see that each nonzero eigenvalue λ of $a_1M_1 + \cdots + a_nM_n$ gives rise to a zero of p_a by the above defined rule. We also see that each zero of p_a arises in this way, since 0 is not such a zero. Taking the derivative with respect to λ in the above equality we see that also the multiplicity of λ as a zero of c_a coincides with the multiplicity of $-\frac{1}{\lambda}$ as a zero of p_a. □

As an immediate corollary we get:

Corollary 2.1.9. *Let $p = \det(\mathcal{M})$ be the determinant of some linear matrix polynomial. Then p is a real zero polynomial.*

Proof. Since hermitian matrices have only real eigenvalues, the claim follows directly from Lemma 2.1.8. □

Also straightforward is the following:

Corollary 2.1.10. *Let $\mathcal{M}$ be a linear matrix polynomial. Then the spectrahedron $\mathcal{S}(\mathcal{M})$ can be recovered from $p = \det(\mathcal{M})$ only. Indeed one has*

$$\mathcal{S}(\mathcal{M}) = \mathcal{R}(p).$$

Proof. $\mathcal{M}(a)$ is positive semidefinite if and only if all eigenvalues of the matrix $a_1M_1+\cdots+a_nM_n$ are greater or equal to -1. By Lemma 2.1.8, this translates to p_a having no zeros in $[0, 1)$. □

Combining Corollary 2.1.9 and Corollary 2.1.10, we finally see:

Corollary 2.1.11. *Each spectrahedron is rigidly convex.*

A deep and important result is that the converse of Corollary 2.1.11 holds in dimension two. We just state it here, and will go into more details in Chapter 3.

Theorem 2.1.12 (Helton & Vinnikov [21])**.** *Each rigidly convex set in $\mathbb{R}^2$ is a spectrahedron.*

We now observe that the set on the left in Figure 2.2 is a spectrahedron, although we are maybe not able to find a defining linear matrix polynomial right away. We also see that the set on the right is not a spectrahedron.

All in all, the question to characterize spectrahedra can be considered as completely solved in the plane. Given a convex set S, one has to check whether there is a real zero polynomial p with $S = \mathcal{R}(p)$. The canonical choice for such p is a polynomial that has as its real zero set the real Zariski closure of the boundary of S.

In higher dimensions, we have the following important open problem:

Question 2.1.13. Is it true that every rigidly convex set in $\mathbb{R}^n$ is a spectrahedron?

Although we cannot answer that question, we will deal with related problems in Chapter 3.

2.2 The Multivariate Hermite Matrix

An interesting tool for checking whether a polynomial is real zero is the *multivariate Hermite matrix*. It is a straightforward generalization of the univariate Hermite matrix. Our approach is from Netzer, Plaumann and Thom [39]. A similar notion appears earlier in Henrion [22], and in several unpublished presentations of Parrilo.

Definition 2.2.1. Let $p \in \mathbb{R}[t]$ be a monic univariate polynomial of degree d and let $\lambda_1, \dots, \lambda_d \in \mathbb{C}$ be all of its zeros. Then for any $k \in \mathbb{N}$, the *k-th Newton sum* $N_k(p)$ is the k-th power sum of the zeros:

$$N_k(p) = \sum_{i=1}^{d} \lambda_i^k.$$

Note that the Newton sums are symmetric functions of the λ_i, and can be expressed as polynomials in the elementary symmetric functions, and thus in the coefficients of the polynomial p. So we need not know the zeros of p to be able to compute its Newton sums. If $p = t^d + p_1 t^{d-1} + p_2 t^{d-2} + \cdots + p_d$, the following well-known formula of Newton allows to recursively obtain these representations:

$$N_k(p) = -\sum_{i=1}^{k-1} p_{k-i} N_i(p) - k p_k, \qquad \text{for } k \geq 1.$$

Here we set $p_i = 0$ for $i > d$. The first few formulas thus look like this:

$$N_0(p) = d, N_1(p) = -p_1, N_2 = p_1^2 - 2p_2$$

$$N_3(p) = -p_1^3 + 3p_1p_2 - 3p_3.$$

Definition 2.2.2. Let $p \in \mathbb{R}[t]$ be a monic univariate polynomial of degree d. The *Hermite matrix* of p is defined as

$$H(p) = (N_{i+j-2}(p))_{i,j=1,\ldots,d} = \begin{pmatrix} N_0(p) & N_1(p) & N_2(p) & \cdots & N_{d-1}(p) \\ N_1(p) & N_2(p) & \cdots & & N_d(p) \\ N_2(p) & \vdots & & & \vdots \\ \vdots & & & & \\ N_{d-1}(p) & N_d(p) & \cdots & & N_{2d-2}(p) \end{pmatrix}.$$

Example 2.2.3. For a quadratic polynomial $p = t^2 + p_1t + p_2$ we find

$$H(p) = \begin{pmatrix} 2 & -p_1 \\ -p_1 & p_1^2 - 2p_2 \end{pmatrix}.$$

What makes the Hermite matrix useful for our purpose is the following classical result, stated for example as Theorem 4.59 in Basu, Pollack and Roy [5].

Proposition 2.2.4. *The polynomial p has only real roots if and only if $H(p)$ is positive semidefinite. $H(p)$ is positive definite if and only if all these real roots are distinct.*

We now need a generalization of the Hermite matrix for a multivariate polynomial $p \in \mathbb{R}[\underline{x}]$. We assume $p(0) = 1$ and write

$$p = 1 + p_1 + \cdots + p_d$$

as a sums of its homogeneous terms. Let $\tilde{p}$ denote the homogenization of p, i.e.

$$\tilde{p} = t^d \cdot p\left(\frac{x}{t}\right) = t^d + p_1t^{d-1} + \cdots + p_d.$$

Definition 2.2.5. The *multivariate Hermite matrix* $\mathcal{H}(p)$ of p is the Hermite matrix of $\tilde{p}$ as a polynomial in t:

$$\mathcal{H}(p) = H(\tilde{p}).$$

We immediately observe that $\mathcal{H}(p)$ is a Hankel matrix, whose entries are polynomial expressions in the homogeneous terms p_i of p. The (i,j)-entry is a homogeneous polynomial in $\underline{x}$ of degree $i+j-2$. From Proposition 2.2.4 we get the following:

Proposition 2.2.6. *The polynomial p is a real zero polynomial if and only if $\mathcal{H}(p)$ is positive semidefinite at each point:*

$$\mathcal{H}(p)(a) \succeq 0 \textit{ for all } a \in \mathbb{R}^n.$$

The polynomial p is smooth if and only if $\mathcal{H}(p)$ is positive definite at each point $a \in \mathbb{R}^n \setminus \{0\}$.

Proof. For $a \in \mathbb{R}^n$, $\mathcal{H}(p)(a)$ is the univariate Hermite matrix of the polynomial

$$t^d + p_1(a)t^{d-1} + \cdots + p_d(a).$$

This is however just the opposite polynomial of

$$p_a(t) = 1 + p_1(a)t + \cdots + p_d(a)t^d.$$

Since the roots of p_a arise from those of the opposite polynomial by the rule $\lambda \mapsto \frac{1}{\lambda}$ (including possible roots at infinity of p_a), the result is clear from Proposition 2.2.4. □

Remark 2.2.7. If a real zero polynomial p of degree d is considered as a polynomial of degree $d' > d$, then the size of the multivariate Hermite matrix increases. The condition of being positive semidefinite at each point is not affected however. On the other hand, the Hermite matrix might now fail to be positive definite at each point, even if it was before. This is clear, since considering p as a polynomial of degree d' means adding zeros at infinity, which might destroy smoothness of p (compare to Definition 2.1.2). If not stated otherwise, we will always consider p as a polynomial of degree $d = \deg(p)$ from now on.

In the case of two variables, the Hermite matrix of a real zero polynomial turns out to be even a sum of squares of polynomial matrices. This was first observed by Parrilo and Henrion. Checking whether a polynomial matrix is a sum of squares can be transformed into a semidefinite programming problem, and thus solved efficiently. So checking whether a bivariate polynomial is real zero can be done efficiently. We give a proof of the result on sums of squares decompositions of the Hermite matrix:

Proposition 2.2.8. *Let $p \in \mathbb{R}[x_1, x_2]$ be a real zero polynomial of degree d. Then*

$$\mathcal{H}(p) = Q^t Q$$

for some $Q \in \mathrm{M}_{2d\times d}\left(\mathbb{R}[x_1, x_2]\right)$.

Proof. Consider $H' := \mathcal{H}(p)(1, x_2) \in \mathrm{Sym}_d(\mathbb{R}[x_2])$. Since this matrix is univariate and positive semidefinite at each point, by Jakubovič [24] it can be written as a square:

$$H' = P^t P, \quad P \in \mathrm{M}_{2d\times d}\left(\mathbb{R}[x_2]\right).$$

We know that the (i, j)-entry of H' is a polynomial of degree at most $i+j-2$. So each entry in the i-th column of P is of degree at most $i - 1$.

Now let $Q \in \mathrm{M}_{2d\times d}\left(\mathbb{R}[x_1, x_2]\right)$ be the matrix arising by homogenizing the i-th column of P to degree $i-1$, i.e. replacing each entry $P_{r,i}$ by $x_1^{i-1} \cdot P_{r,i}(\frac{x_2}{x_1})$. Now the (i, j)-entry of $Q^t Q$ is the homogenization to degree $i + j - 2$ of the (i, j)-entry of H'. This equals the (i, j)-entry of $\mathcal{H}(p)$. □

We will see in Chapter 3 how sums of squares decompositions of the Hermite matrix are closely related to determinantal representations of the polynomial p.

We finish this section with a first observation on the topology of the set of real zero polynomials. For $n, d \in \mathbb{N}$ we denote by $\mathcal{R}_{n,d}$ the set of all real zero polynomials of degree at most d in n variables. It is a subset of $\mathbb{R}[\underline{x}]_d$, the finite dimensional space of all polynomials of degree at most d. By definition, $\mathcal{R}_{n,d}$ is contained in the hyperplane defined by the condition $p(0) = 1$.

Proposition 2.2.9. *The set $\mathcal{R}_{n,d}$ is a closed semi-algebraic set. Inside of the hyperplane of $\mathbb{R}[\underline{x}]_d$ defined by $p(0) = 1$ it has nonempty interior. The interior consists precisely of those real zero polynomials that are smooth, when considered as polynomials of degree d. Furthermore, $\mathcal{R}_{n,d}$ is semi-algebraically connected.*

Proof. Being a real zero polynomial can be expressed in a formula of first order logic, using quantifiers. By quantifier elimination of the theory of real closed fields, the set $\mathcal{R}_{n,d}$ is a semi-algebraic subset of $\mathbb{R}[\underline{x}]_d$.

From the fact that p is a real zero polynomial if and only if $\mathcal{H}(p)$ is positive semidefinite at each point, and since the Hermite matrix depends continuously on the polynomial, it is clear that $\mathcal{R}_{n,d}$ is closed. Now assume that p is smooth, as a polynomial of degree d. Then its Hermite matrix is positive definite, at each point $a \in S^{n-1}$. This is clearly an open condition, since the sphere S^{n-1} is compact. Assume conversely that p is not smooth. We show that p does lie in the boundary of $\mathcal{R}_{n,d}$, inside of the hyperplane defined by $p(0) = 1$. If some p_a has a multiple zero which does not lie at infinity, it is easy to see that there are polynomials q arbitrary close to p, with $q(0) = 1$ and q_a having a non-real root. One can in fact reduce to the univariate case. So now assume that p_a has a multiple root at infinity, for some $a \neq 0$. This just means $\deg(p_a) \leq d - 2$. We can assume that a is the first coordinate direction e_1. Write $p = 1 + p_1 + \cdots + p_d$ as a sum of homogeneous terms. Then set

$$p^{(\varepsilon)} := p + \varepsilon \cdot x_1^2 \cdot (1 + p_1 + \cdots + p_{d-2}).$$

Clearly $p^{(\varepsilon)}$ is of degree at most d, fulfills $p^{(\varepsilon)}(0) = 1$ and converges to p, for $\varepsilon \to 0$. We find

$$p_a^{(\varepsilon)} = p_a + \varepsilon \cdot t^2 \cdot p_a = p_a \cdot (1 + \varepsilon \cdot t^2),$$

and this univariate polynomial has a non-real zero for $\varepsilon > 0$.

Finally, every real zero polynomial p can be connected to the polynomial 1 by the continuous semi-algebraic path $s \mapsto p(s \cdot \underline{x})$, where $s \in [0, 1]$. □

2.3 Hyperbolic Polynomials

As already mentioned, there is a strong connection between real zero polynomials and hyperbolic polynomials. Hyperbolic polynomials play an important role in some areas of partial differential equations, for example. We start with the definition.

Definition 2.3.1. Let $q \in \mathbb{R}[x_0, x_1, \ldots, x_n]$ be a homogeneous polynomial. Then q is *hyperbolic* if $q(1, 0, \ldots, 0) = 1$ and for all $(a_0, \ldots, a_n) \in \mathbb{R}^{n+1}$, the univariate polynomial

$$q(t) = q(a_0 + t, a_1, \ldots, a_n)$$

has only real roots.

The connection to real zero polynomials is subsumed in the following easy and well-known lemma.

Lemma 2.3.2. *If $p \in \mathbb{R}[x_1, \ldots, x_n]$ is a real zero polynomial, then its homogenization*

$$\widetilde{p} = x_0^{\deg(p)} p\left(\frac{x}{x_0}\right)$$

is hyperbolic. Conversely, if $q \in \mathbb{R}[x_0, \ldots, x_n]$ is hyperbolic, then its dehomogenization

$$p = q(1, x_1, \ldots, x_n)$$

is a real zero polynomial.

Proof. First note that $\widetilde{p}(1, 0, \ldots, 0) = p(0) = 1$. Now let $(a_0, \ldots, a_n) \in \mathbb{R}^{n+1}$ and $\lambda \in \mathbb{C}$, and assume

$$\widetilde{p}(a_0 + \lambda, a_1, \ldots, a_n) = 0.$$

In case that $a_0 + \lambda = 0$, λ is clearly real. In the other case we find

$$\begin{aligned} 0 &= \left(\frac{1}{a_0 + \lambda}\right)^{\deg \widetilde{p}} \cdot \widetilde{p}(a_0 + \lambda, a_1, \ldots, a_n) \\ &= \widetilde{p}\left(1, \frac{a_1}{a_0 + \lambda}, \ldots, \frac{a_n}{a_0 + \lambda}\right) \\ &= p\left(\frac{a_1}{a_0 + \lambda}, \ldots, \frac{a_n}{a_0 + \lambda}\right). \end{aligned}$$

Since p is assumed to be real zero, this implies $\frac{1}{a_0+\lambda} \in \mathbb{R}$, and thus also $\lambda \in \mathbb{R}$. So $\widetilde{p}$ is hyperbolic.

For the second claim first observe $p(0) = q(1, 0) = 1$. Now let $(a_1, \ldots, a_n) \in \mathbb{R}^n$ and $\lambda \in \mathbb{C}$ with

$$0 = p(\lambda \cdot a) = q(1, \lambda \cdot a_1, \ldots, \lambda \cdot a_n).$$

Since $\lambda \neq 0$ and q is homogeneous, this implies

$$q\left(\frac{1}{\lambda}, a_1, \ldots, a_n\right) = 0,$$

and since q is hyperbolic, $\frac{1}{\lambda}$ and thus λ is real. So p is a real zero polynomial. $\square$

There is another way to get a real zero polynomial from a hyperbolic polynomial, besides dehomogenization. The idea appears first in Brändén's paper [8], the proof is as easy as the last one.

Remark 2.3.3. If $q \in \mathbb{R}[x_0, \dots, x_n]$ is hyperbolic, then the shifted polynomial

$$p(x_0, \dots, x_n) := q(x_0 + 1, x_1, \dots, x_n)$$

is a real zero polynomial.

There is an analog of the rigidly convex set for hyperbolic polynomials. It is called a *hyperbolicity cone*:

Definition 2.3.4. Let $q \in \mathbb{R}[x_0, \dots, x_n]$ be hyperbolic. Then the set

$$\Lambda(q) := \left\{(a_0, \dots, a_n) \in \mathbb{R}^{n+1} \mid q(a_0 + t, a_1, \dots, a_n) \text{ has no positive root}\right\}$$

is called the *hyperbolicity cone* defined by q.

If p is a real zero polynomial, and $\widetilde{p}$ is its homogenization, then the following fact is easily checked:

$$\mathcal{R}(p) = \Lambda(\widetilde{p}) \cap \{x_0 = 1\}.$$

As already mentioned, Gårding [16] has shown hyperbolicity cones to be convex cones. So we see that rigidly convex sets are convex. This can also be deduced from Theorem 2.1.12, using the fact that spectrahedra are obviously convex, and that convexity can be checked on two-dimensional subspaces. Renegar [49] has shown hyperbolicity cones to be basic closed semi-algebraic. So the same is true for rigidly convex sets. He also showed that all faces of hyperbolicity cones are exposed. Again this is true for rigidly convex sets, as noted in Netzer, Plaumann and Schweighofer [38].

Corollary 2.3.5. *The faces of a rigidly convex set are exposed.*

Proof. Let $S = \mathcal{R}(p) \subseteq \mathbb{R}^n$ be rigidly convex. Let $\widetilde{p}$ be the homogenization of p, and denote by $C = \Lambda(\widetilde{p})$ the hyperbolicity cone of $\widetilde{p}$ in $\mathbb{R}^{n+1}$. Let F_0 be a face of S. For any two points $a \neq b \in \mathbb{R}^{n+1}$, let $g(a, b)$ denote the line passing through a and b. Take a_0 in the relative interior of F_0, and let F be the set of all points $c \in C$ such that a_0 lies in the relative interior of $g(c, a_0) \cap C$ (one has to include a_0 to F as well). One checks that F is a face of C and that

$$F \cap \{x_0 = 1\} = F_0.$$

Since F is exposed as a subset of C, so is F_0 as a subset of S. $\square$

In the next section of this chapter, we introduce the so-called *Renegar derivative* of hyperbolic and real zero polynomials. These derivatives were the main tools in proving Renegar's results.

2.4 Renegar Derivatives

Definition 2.4.1. Let $q \in \mathbb{R}[x_0, x_1, \ldots, x_n]$ be a hyperbolic polynomial. Then

$$\partial_R(q) := \frac{1}{\deg(q)} \cdot \frac{\partial q}{\partial x_0}$$

is called the Renegar derivative of q. Iteratively, one defines

$$\partial_R^{(i+1)}(q) := \partial_R\left(\partial_R^{(i)}(q)\right).$$

From Rolle's Theorem it immediately follows that $\partial_R(q)$ is again hyperbolic, and that

$$\Lambda(q) \subseteq \Lambda(\partial_R(q))$$

holds. To prove his results, Renegar defined for $a = (a_0, \ldots, a_n) \in \Lambda(q)$

$$\mathrm{mult}(a)$$

to be the multiplicity of 0 as a zero of the polynomial $q(a_0 + t, a_1, \ldots, a_n)$. He showed that if $\mathrm{mult}(a) = m > 0$, then a is a boundary point of

$$\Lambda\left(\partial_R^{(m-1)}(q)\right)$$

and a regular point of the polynomial $\partial_R^{(m-1)}(q)$. The tangent space of $\partial_R^{(m-1)}(q)$ at a then exposes the smallest face of $\Lambda(q)$ containing a. He also showed that the finite sequence of iterated Renegar derivatives of q define $\Lambda(q)$ as a basic closed semi-algebraic set, i.e.

$$\Lambda(q) = \left\{a \in \mathbb{R}^{n+1} \mid \partial_R^{(i)}(q)(a) \geq 0 \text{ for all } i = 0, \ldots, \deg(q) - 1\right\}.$$

We translate the same notion into the setup of real zero polynomials.

Definition 2.4.2. Let $p \in \mathbb{R}[\underline{x}]$ be a real zero polynomial and let

$$\widetilde{p} = x_0^{\deg(p)} \cdot p\left(\frac{x}{x_0}\right)$$

denote its homogenization. Then

$$\partial_R(p) := \partial_R(\widetilde{p})(1, \underline{x})$$

is called the *Renegar derivative* of p. Again, we define

$$\partial_R^{(i+1)}(p) := \partial_R\left(\partial_R^{(i)}(p)\right)$$

inductively.

Remark 2.4.3. The Renegar derivative of a real zero polynomial can also be defined as follows. Write $p = 1 + p_1 + \cdots + p_d$ as a sum of its homogeneous components. Then

$$\partial_R(p) = 1 + \frac{d-1}{d} \cdot p_1 + \cdots + \frac{2}{d} \cdot p_{d-2} + \frac{1}{d} \cdot p_{d-1} = \sum_{i=0}^{d-1} \frac{d-i}{d} \cdot p_i.$$

It should now be clear from the facts of the last section that $\partial_R(p)$ is again a real zero polynomial. For each direction $a \in \mathbb{R}^n$, the zeros of $\partial_R(p)$ lie in between the zeros of p (including possible zeros at infinity). The multiplicities of zeros of p in direction a are thus reduced by one. Furthermore, one has

$$\mathcal{R}(p) \subseteq \mathcal{R}(\partial_R(p)),$$

and the finite sequence of Renegar derivatives define $\mathcal{R}(p)$ as a basic closed semi-algebraic set, i.e.

$$\mathcal{R}(p) = \left\{a \in \mathbb{R}^n \mid \partial_R^{(i)}(p)(a) \geq 0 \text{ for all } i = 0, \ldots, \deg(p) - 1\right\}.$$

One defines $\operatorname{mult}(a)$ for $a \in \mathcal{R}(p)$ as the multiplicity of 1 as a zero of the univariate polynomial $p_a(t)$. If $\operatorname{mult}(a) = m > 0$, then a is a boundary point of

$$\mathcal{R}\left(\partial_R^{(m-1)}(p)\right)$$

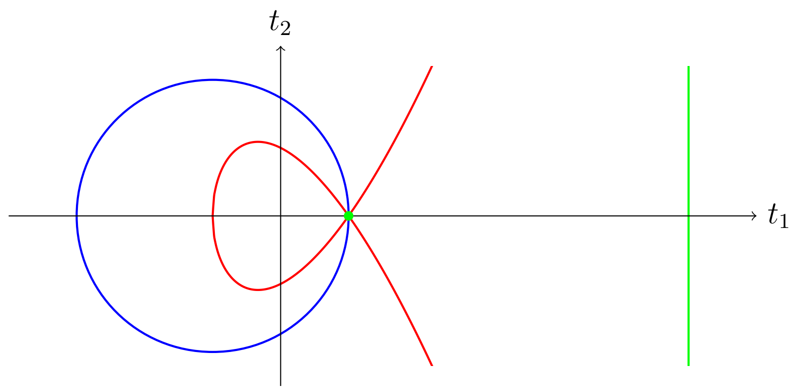
and a regular point of $\partial_R^{(m-1)}(p)$. The tangent space of $\partial_R^{(m-1)}(p)$ at a exposes the smallest face of $\mathcal{R}(p)$ containing a.

Example 2.4.4. In Figure 2.3 you can see the zero set of

$$p = x_1^3 - x_1^2 - x - x_2^2 + 1$$

again, and the zero sets of its first two Renegar derivatives. The green dot is a point in $\mathcal{R}(p)$ of multiplicity 2, and it is exposed by the tangent space to the blue curve at that point.

Figure 2.3:



2.5 Smooth Approximation

In this section we describe a method for smooth approximation of real zero polynomials. It is a straightforward translation of a result of Nuij [42] for hyperbolic polynomials. Recall that we call a real zero polynomial $p \in \mathbb{R}[\underline{x}]$ *smooth*, if none of the polynomials p_a with $a \in \mathbb{R}^n \setminus \{0\}$ has a multiple root, including possible roots at infinity. This is equivalent to the Hermite matrix $\mathcal{H}(p)$ being positive definite at each point $a \neq 0$.

Let $p \in \mathbb{R}[\underline{x}]$ be a real zero polynomial. For $\varepsilon > 0$ and $i \in \{1, \ldots, n\}$ we define

$$G_{\varepsilon,i}(p) := p + \varepsilon \cdot x_i \cdot \partial_R(p).$$

Here, $\partial_R(p)$ is the Renegar derivative of p. Note that for small ε, $G_{\varepsilon,i}(p)$ has the same degree as p. We then define

$$G_\varepsilon(p) := G_{\varepsilon,1} \cdots G_{\varepsilon,n}(p).$$

Lemma 2.5.1. *If p is a real zero polynomial, then $G_\varepsilon(p)$ is again real zero. If p_a has a zero λ of multiplicity m, for some $a \in \mathbb{R}^n$, then λ is a zero of $G_\varepsilon(p)_a$ of multiplicity at most $m-1$. The zeros of $G_\varepsilon(p)_a$ that do not arise in this way are all simple.*

Proof. Consider the homogenization $\widetilde{p} = x_0^d \cdot p\left(\frac{x}{x_0}\right)$ and its hyperbolic Renegar derivative $\partial_R(\widetilde{p}) = \frac{1}{d} \cdot \frac{\partial \widetilde{p}}{\partial x_0}$. The polynomial

$$q_i = \widetilde{p} + \varepsilon \cdot x_i \cdot \partial_R(\widetilde{p})$$

is homogeneous, for all $i = 1, \ldots, n$. Consider now for $a \in \mathbb{R}^n$ the univariate polynomial

$$q_i(x_0, a) = \widetilde{p}(x_0, a) + \varepsilon \cdot a_i \cdot \partial_R(\widetilde{p})(x_0, a).$$

As described by Nuij [42], for a univariate polynomial h with only real zeros, and $s \neq 0$, the polynomial $h + sh'$ also has only real roots, and reduced zero multiplicities as desired for our lemma. So each q_i is hyperbolic, and the zero multiplicities of $q_i(x_0, a)$ are reduced as desired, if $a_i \neq 0$. As seen in the proof of Lemma 2.3.2, the polynomial

$$G_{\varepsilon,i}(p) = p + \varepsilon \cdot x_i \cdot \partial_R(p)$$

is real zero and has the desired zero multiplicity reduction in direction of a, if $a_i \neq 0$. Since we repeat this procedure for all $i = 1, \ldots, n$, this proves the claim. □

We denote by $G_\varepsilon^m(p)$ the m-fold application of the operator G_ε to p. The following is now clear.

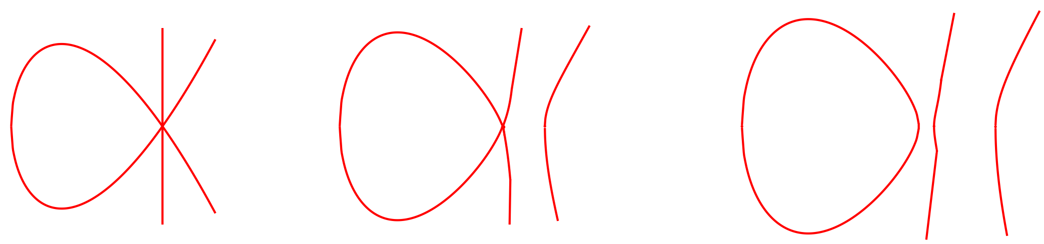
Corollary 2.5.2. *If m is the highest multiplicity of a zero of p_a among all directions $a \neq 0$, then $G_\varepsilon^{m-1}(p)$ is smooth. The polynomials $G_\varepsilon^{m-1}(p)$ are all real zero polynomials, of the same degree as p if ε is small enough, and they converge to p for $\varepsilon \to 0$.*

Example 2.5.3. In Figure 2.4 you see, from left to right, the zero sets of $p, G_\varepsilon(p)$ and $G_\varepsilon^2(p)$, for

$$p = (1 - x_1) \cdot (x_1^3 - x_1^2 - x_1 - x_2^2 + 1)$$

and $\varepsilon = 1/5$.

Figure 2.4:



Recall that we denote by $\mathcal{R}_{n,d}$ the set of all real zero polynomials in n variables of degree d. We have seen in Proposition 2.2.9 that $\mathcal{R}_{n,d}$ is closed semi-algebraic, connected, and of non-empty interior in the hyperplane of $\mathbb{R}[\underline{x}]_d$ defined by $p(0) = 1$. The following is Nuij's result, translated to real zero polynomials.

Proposition 2.5.4. *$\mathcal{R}_{n,d}$ is regular in the hyperplane defined by $p(0) = 1$, i.e. it is the closure of its interior.*

Proof. Let $p \in \mathcal{R}_{n,d}$. If $\deg(p) < d$, we can clearly approximate p with real zero polynomials of degree d. For example take $p \cdot q(\varepsilon \cdot \underline{x})$, with a suitable real zero polynomial q. Now if $\deg(p) = d$, we can approximate p by smooth real zero polynomials of degree d, as shown in Corollary 2.5.2. But smooth real zero polynomials of degree d belong to the interior of $\mathcal{R}_{n,d}$ in the hyperplane defined by the condition $p(0) = 1$, as was shown in Proposition 2.2.9. $\square$

Remark 2.5.5. One could also use the simpler approximation

$$p + \varepsilon \cdot \partial_R(p)$$

for a real zero polynomial p and $\varepsilon > 0$. This is again a real zero polynomial, as one shows similar as in the proof of Lemma 2.5.1, using this time that for $\varepsilon > 0$

$$h + \varepsilon \cdot t \cdot h'$$

splits over $\mathbb{R}$, if $h \in \mathbb{R}[t]$ does. This approximation does however not smooth zeros at infinity, as one checks for example for $p = 1 - x_1^2 \in \mathbb{R}[x_1, x_2]$.

Chapter 3

Determinantal Representations

3.1 Preliminaries

Recall that a central issue of our work is the characterization of spectrahedra. As we have explained in the last chapter, each spectrahedron is rigidly convex, and we are interested in the question whether the converse is true. One of the crucial observations in the last chapter was Corollary 2.1.10. It says that the spectrahedron $\mathcal{S}(\mathcal{M})$, defined by the linear matrix polynomial $\mathcal{M}$, can be recovered from the real zero determinantal polynomial

$$p = \det \mathcal{M}.$$

In fact we found

$$\mathcal{S}(\mathcal{M}) = \mathcal{R}(p)$$

to hold. So for a rigidly convex set $\mathcal{R}(p)$ to be a spectrahedron, it is clearly enough that p can be realized as the determinant of a linear matrix polynomial $\mathcal{M}$. Recall at this point that we assume all linear matrix polynomials to be monic. Helton and Vinnikov proved their already stated main result (Theorem 2.1.12 above) via the following stronger theorem:

Theorem 3.1.1 (Helton & Vinnikov [21])**.** *Let $p \in \mathbb{R}[x_1, x_2]$ be a real zero polynomial of degree d. Then there is a symmetric linear matrix polynomial $\mathcal{M}$ of size d with*

$$p = \det \mathcal{M}.$$

Note that Sturmfels, Plaumann and Vinzant [44] describe several ways of how to construct the matrix polynomial $\mathcal{M}$ for a given polynomial p.

Helton and Vinnikov already note that one cannot expect the same result to be true in higher dimensions. By counting parameters, one sees that the set of all n-tuples of symmetric matrices of size d is of dimension

$$n \cdot \binom{d+1}{2}.$$

The semi-algebraic set of real zero polynomials in n variables of degree d is of dimension

$$\binom{d+n}{d} - 1$$

however (see also Section 3.3.1 below). So one cannot expect the analog of Theorem 3.1.1 to hold for $n \geq 3$. Also passing to hermitian matrices instead of symmetric matrices does not solve the problem of dimensional differences. Helton and Vinnikov have however conjectured that allowing for matrices of a larger size than the degree d would make the result valid in higher dimensions. This is however also not true, as was shown first by Brändén [8], and will be discussed below in more detail.

If we are only interested in spectrahedra however, the Helton-Vinnikov Theorem would suffice to hold in an even weaker version. Indeed representing some *multiple* qp as a determinant, as long as $\mathcal{R}(qp) = \mathcal{R}(p)$ holds, proofs that $\mathcal{R}(p)$ is a spectrahedron. In particular, representing some power p^r of p would do. Although Brändén showed that this is also not possible in general, some positive results can be proven. We will discuss positive and negative results concerning determinantal representations of real zero polynomials in the following sections.

3.2 Bounds on the Size

One cannot expect to realize every real zero polynomial of degree d as the determinant of some linear matrix polynomial of size d. But one can look for matrix polynomials of larger size of course. There can indeed by degree cancellation when computing determinants, as the following example shows.

Example 3.2.1. Let $p_n = 1 - x_1^2 - \cdots - x_n^2 \in \mathbb{R}[\underline{x}]$. Then

$$p_n = \det \begin{pmatrix} 1 & x_1 & \cdots & x_n \\ x_1 & 1 & & \\ \vdots & & \ddots & \\ x_n & & & 1 \end{pmatrix}.$$

The linear matrix polynomial occuring here is of size $n+1$, whereas p_n has degree 2.

We will prove lower and upper bounds on the possible size of a linear matrix polynomial in the following. The results are published in Netzer and Thom [41]. Most of the results are based on the following easy lemma.

Lemma 3.2.2. *Let $\mathcal{M} = I + x_1M_1 + \cdots + x_nM_n$ be a linear matrix polynomial and assume $p = \det \mathcal{M}$ is of degree d. Then each matrix in the real vector space*

$$\operatorname{span}_{\mathbb{R}} \{M_1, \ldots, M_n\}$$

has rank at most d, and the generic linear combination has rank precisely d.

Proof. The rank of any matrix $a_1M_1 + \cdots + a_nM_n$ is the number of its nonzero eigenvalues, which by Lemma 2.1.8 correspond to the zeros of the univariate polynomial p_a. Now each p_a has degree at most d, and thus at most d zeros. For all a for which p_a has degree precisely d, the matrix is of rank precisely d. This is true for the generic choice of a. □

3.2.1 Upper Bounds

There is a surprisingly simple to prove upper bound on the size of a linear matrix polynomial, depending on the degree of its determinant and the number of variables.

Theorem 3.2.3. *Let $p \in \mathbb{R}[x_1, \ldots, x_n]$ be a real zero polynomial of degree d. If p has a symmetric/hermitian determinantal representation, then it has a symmetric/hermitian determinantal representation of size nd.*

Proof. Assume

$$p = \det (I + x_1M_1 + \ldots + x_nM_n)$$

for some matrices $M_i \in \mathrm{H}_k(\mathbb{C})$. Let $K_i \subseteq \mathbb{C}^k$ be the kernel of the linear map defined by M_i. By Lemma 3.2.2 we find $\dim_{\mathbb{C}} K_i \geq k-d$ for all i. From the dimension formula for subspaces we get

$$\dim\left(K_1 \cap \ldots \cap K_n\right) \geq k - nd.$$

So if $k > nd$ we can simultaneously split off a $k-nd$ block of zeros of each M_i, by conjugation with a unitary matrix. This produces a determinantal representation of p of size nd. The same argument works with symmetric matrices and an orthogonal base change. □

This last result will allow an easy dimension count argument to show that indeed almost no real zero polynomial admits a determinantal representation. See Section 3.3 for that result.

Under an additional geometric condition one can decrease the upper bound of nd to d. We need the following technical proposition.

Proposition 3.2.4. *Let $V \subseteq \mathrm{H}_k(\mathbb{C})$ be an $\mathbb{R}$-subspace of hermitian matrices, such that all elements of V have rank at most d. If V contains a positive semidefinite matrix of rank d, then there is some unitary matrix $Q \in M_k(\mathbb{C})$ such that*

$$Q^*VQ \subseteq \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 0 \end{array}\right) \mid A \in \mathrm{H}_d(\mathbb{C}) \right\}.$$

If $V \subseteq \mathrm{Sym}_k(\mathbb{R})$, then Q can be chosen real orthogonal.

Proof. After a unitary/orthogonal change of coordinates we can assume that V contains a matrix

$$P' = \left(\begin{array}{c|c} P & 0 \\ \hline 0 & 0 \end{array}\right)$$

where P is a positive definite matrix of size d. Let A' be an arbitrary matrix from V and write

$$A' = \left(\begin{array}{c|c} A & B \\ \hline B^* & C \end{array}\right)$$

We show $B = 0$ and $C = 0$.

We know that $A' + \lambda P'$ has rank at most d for all $\lambda \in \mathbb{R}$, and the upper left block of size d in this matrix has arbitrary large eigenvalues, for λ big enough. Consider any quadratic submatrix of size $d+1$ of $A' + \lambda P'$, containing this upper left block, obtained by deleting the same set of rows and columns:

$$\left(\begin{array}{c|c} A+\lambda P & b \\ \hline b^* & c \end{array}\right)$$

Here $b \in \mathbb{C}^d$ is a certain column of B, and c is the corresponding diagonal entry of C. From the rank condition we see that the last column in this matrix is a linear combination of the first d columns, at least for $\lambda \in \mathbb{R}$ large enough. If $v = (v_1, \ldots, v_d)^t$ is the vector of coefficients of this linear combination, we have

$$(A+\lambda P)v = b \text{ and } b^* v = c,$$

which implies $v^*(A+\lambda P)v = \overline{c} = c$. This means that for large values of λ, the norm of v must be arbitrary small. But his is only compatible with the condition $b^*v = c$ if $c = 0$. Since $A+\lambda P$ is positive definite, this then implies $v = 0$, and thus $b = 0$. We have now shown $B = 0$, and this implies $C = 0$, using again the rank condition for large values of λ. □

The following theorem gives the best possible upper bound, under an additional geometric condition.

Theorem 3.2.5. *Let $\mathcal{M}$ be a symmetric/hermitian linear matrix polynomial and let d denote the degree of $p = \det \mathcal{M}$. If the spectrahedron defined by $\mathcal{M}$ contains a full-dimensional cone, then p can be realized as the determinant of a symmetric/hermitian linear matrix polynomial of size d.*

Proof. If the whole positive half-ray through some $a \in \mathbb{R}^n$ is contained in the spectrahedron, then $a_1M_1 + \cdots + a_nM_n$ is positive semidefinite. Since the generic linear combination has rank d, there is such a for which $a_1M_1 + \cdots + a_nM_n$ has rank d. Now apply Proposition 3.2.4 to the space spanned by the M_i, to reduce the size of $\mathcal{M}$ to d, without changing the determinant. □

Remark 3.2.6. Brändén proved a special case of Theorem 3.2.5 in his Theorem 2.2 in [8]. He considered real zero polynomials p that arise as shifts of hyperbolic polynomials, as explained in Remark 2.3.3. Note that for these polynomials, $\mathcal{R}(p)$ clearly contains a full-dimensional cone. Brändén's proof relies heavily on the Cauchy-Binet formula.

Remark 3.2.7. Note that the proofs of both Theorem 3.2.3 and Theorem 3.2.5 not only show that there *is* a determinantal representation of relatively small size, but in fact *every* determinantal representation can be reduced in size by a base change. A representation of very large size can thus only arise as a trivial extensions of a relatively small one.

Before we apply these upper bounds to get negative results on determinantal representations, we prove some lower bounds.

3.2.2 Lower Bounds

We will use results on spaces of symmetric matrices of low rank, of which there are plenty in the literature. The following is the main result from Meshulam [33], stated in the terminology of Loewy and Radwan [29].

Theorem 3.2.8. *Let $V \subseteq \mathrm{Sym}_k(\mathbb{R})$ be a linear subspace such that all elements of V have rank at most d. Then*

$$\dim V \leq \alpha(k,d)$$

which computes as follows. For $d = 2e$ even:

$$\alpha(k,d) = \begin{cases} \binom{d+1}{2} & \text{if } 2k \leq 5e+1 \\ \binom{e+1}{2} + e(k-e) & \text{if } 2k > 5e+1. \end{cases}$$

For $d = 2e+1$ odd:

$$\alpha(k,d) = \begin{cases} \binom{d+1}{2} & \text{if } 2k \leq 5(e+1) \\ \binom{e+1}{2} + e(k-e) + 1 & \text{if } 2k > 5(e+1). \end{cases}$$

To be able to apply this result, we note the following easy fact:

Lemma 3.2.9. *Let $\mathcal{M} = I + x_1 M_1 + \cdots + x_n M_n$ be a linear matrix polynomial. If $\mathcal{S}(\mathcal{M})$ does not contain a full line, then $M_1, \ldots, M_n$ are $\mathbb{R}$-linearly independent.*

Proof. Assume that some M_i is an $\mathbb{R}$-linear combination of the other M_j, and replace it by this linear combination. We see that $\mathcal{S}(\mathcal{M})$ is the inverse image under a linear map of some nonempty spectrahedron in $\mathbb{R}^{n-1}$. It thus contains a full line. $\square$

Our first lower bound shows that under a mild compactness assumption, *no* polynomial has a *very small* determinantal representation, if the number of variables is large enough.

Theorem 3.2.10. *Let $\mathcal{M}$ be a symmetric linear matrix polynomial of size k, defining a spectrahedron in $\mathbb{R}^n$ that does not contain a full line. Let d denote the degree of $p = \det \mathcal{M}$ and assume $n > \binom{d+1}{2}$. If d is even then*

$$k \geq \frac{2n}{d} + \frac{d-2}{4},$$

if d is odd then

$$k \geq \frac{2(n-1)}{d-1} + \frac{d-3}{4}.$$

Proof. From Lemma 3.2.2, Theorem 3.2.8 and Lemma 3.2.9 we obtain

$$n \leq \alpha(k, d).$$

The result is now just a straightforward computation. □

Remark 3.2.11. Note that Lemma 3.2.9 immediately implies

$$n \leq \dim \mathrm{Sym}_k(\mathbb{R}) = \binom{k+1}{2}.$$

This however only gives a lower bound for k which depends on the square root of n.

Example 3.2.12. Let $d = 2$. Applying Theorem 3.2.10 shows that if $n > 3$, then $k \geq n$. So for example the real zero polynomial

$$p_n = 1 - x_1^2 - \cdots - x_n^2$$

cannot be realized as the determinant of a symmetric linear matrix polynomial of size smaller than n, except possibly for $n = 3$ (although the argument from the proof of Theorem 3.3.7 below will show that also for $n = 3$ there is no symmetric representation of size 2). There is always a realization of size $n + 1$, as explained in Example 3.2.1.

To obtain similar bounds for hermitian matrices, we need some more preparation.

Lemma 3.2.13. *For $M \in \mathrm{H}_k(\mathbb{C})$ write $M = R + iI$ with a real symmetric matrix R and a real skew-symmetric matrix I. Define*

$$\widetilde{M} = \left(\begin{array}{c|c} R & I \\ \hline -I & R \end{array}\right),$$

a real symmetric matrix of size $2k$. Then $\widetilde{M}$ has the same eigenvalues as M, with double multiplicities.

Proof. Let λ be an eigenvalue of M and $z \in \mathbb{C}^k$ a corresponding eigenvector. If $z = a + ib$ with $a, b \in \mathbb{R}^k$, then

$$Ra - Ib = \lambda a \quad \text{and} \quad Rb + Ia = \lambda b.$$

So both

$$\begin{pmatrix} -a \\ b \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b \\ a \end{pmatrix}$$

are eigenvectors with eigenvalue λ of $\widetilde{M}$. Now let $z_1, \dots, z_m \in \mathbb{C}^k$ be complex vectors and write each $z_j = a_j + ib_j$ with $a_j, b_j \in \mathbb{R}^k$. One checks that $z_1, \dots, z_m$ are $\mathbb{C}$-linearly independent if and only if the vectors

$$\begin{pmatrix} -a_1 \\ b_1 \end{pmatrix}, \begin{pmatrix} b_1 \\ a_1 \end{pmatrix}, \dots, \begin{pmatrix} -a_m \\ b_m \end{pmatrix}, \begin{pmatrix} b_m \\ a_m \end{pmatrix}$$

are $\mathbb{R}$-linearly independent in $\mathbb{R}^{2k}$. This finishes the proof. □

Lemma 3.2.14. *Let $\mathcal{M}$ be a hermitian linear matrix polynomial of size k, and write $\mathcal{M} = \mathcal{R} + i\mathcal{I}$ with real symmetric and skew-symmetric linear matrix polynomials $\mathcal{R}$ and $\mathcal{I}$. Define*

$$\widetilde{\mathcal{M}} := \left(\begin{array}{c|c} \mathcal{R} & \mathcal{I} \\ \hline -\mathcal{I} & \mathcal{R} \end{array}\right).$$

Then $\widetilde{\mathcal{M}}$ is a symmetric linear matrix polynomial of size $2k$ with

$$\det \widetilde{\mathcal{M}} = (\det \mathcal{M})^2.$$

Proof. Write $\widetilde{p} = \det \widetilde{\mathcal{M}}$ and $p = \det \mathcal{M}$. By Lemma 3.2.13, the eigenvalues of $\widetilde{\mathcal{M}}(a)$ are the same as the eigenvalues of $\mathcal{M}(a)$, just with double multiplicity, for each $a \in \mathbb{R}^n$. Lemma 2.1.8 implies that $\widetilde{p}_a$ has the same zeros as p_a, just with double multiplicities, for each $a \in \mathbb{R}^n$. So $\widetilde{p}_a = (p_a)^2$ for all a, which implies $\widetilde{p} = p^2$. □

Note that the last lemma proves Property 4 from Section 1.3, i.e. that each spectrahedron is definable by a symmetric linear matrix polynomial. Indeed for $p = \det \mathcal{M}$ we have

$$\mathcal{S}(\mathcal{M}) = \mathcal{R}(p) = \mathcal{R}(p^2) = \mathcal{S}(\widetilde{\mathcal{M}}).$$

From Lemma 3.2.14 we can now also immediately deduce the following analog of Theorem 3.2.10 for hermitian matrices.

Theorem 3.2.15. *Let $\mathcal{M}$ be a hermitian linear matrix polynomial of size k, defining a spectrahedron in $\mathbb{R}^n$ that does not contain a full line. Let d denote the degree of $p = \det \mathcal{M}$ and assume $n > \binom{2d+1}{2}$. Then*

$$k \geq \frac{n}{2d} + \frac{d-1}{4}.$$

3.2.3 Representations of Hyperbolic Polynomials

We have already explained the close relationship between real zero polynomials and hyperbolic polynomials in Section 2.3. We want to add a short comparison between determinantal representations of both kinds of polynomials. First note that if $M_1, \ldots, M_n \in \mathrm{Her}_k(\mathbb{C})$, then

$$q = \det\left(x_0 I + x_1 M_1 + \cdots + x_n M_n\right)$$

is a hyperbolic polynomial. This follows again from the fact that hermitian matrices have only real eigenvalues. So as for real zero polynomials, one can ask whether a hyperbolic polynomial has such a determinantal representation.

Now note that there is a slight but significant difference to the case of real zero polynomials. Namely, if $M_1, \ldots, M_n \in \mathrm{Her}_k(\mathbb{C})$, then the function

$$(x_0, x_1, \ldots, x_n) \mapsto \det\left(x_0 I + x_1 M_1 + \cdots + x_n M_n\right)$$

is homogeneous of degree k. That means the polynomial

$$q = \det\left(x_0 I + x_1 M_1 + \cdots + x_n M_n\right)$$

is necessarily of degree k as well. So in contrast to the case of real zero polynomials, there can be *no degree cancellation* when computing the determinant. An easy count of parameters thus shows immediately that not every hyperbolic polynomial can have a determinantal representation as above.

Now let $M_1, \ldots, M_n \in \mathrm{Her}_k(\mathbb{C})$ be hermitian matrices and assume that $p = \det\left(I + x_1 M_1 + \cdots + x_n M_n\right)$ is of degree d. Denote by

$$\widetilde{p} = x_0^d \cdot p\left(\frac{x}{x_0}\right)$$

the homogenization of p, which is a hyperbolic polynomial. We then find

$$\det\left(x_0 I + x_1 M_1 + \cdots + x_n M_n\right) = x_0^{k-d} \cdot \widetilde{p},$$

i.e. the homogenized matrix polynomial has as its determinant the homogenization of p *up to a power of* x_0. The power is exactly the difference of the matrix size k and the degree d of p.

Conversely, if the hyperbolic polynomial $\widetilde{p}$ has a hermitian determinantal representation, it has one of size d. Setting $x_0 = 1$ gives a determinantal representation of p of size d. We summarize this in the following observation:

Observation 3.2.16. *Representing hyperbolic polynomials as determinants of homogeneous linear matrix polynomials is exactly the same as trying to represent real zero polynomials as determinants of linear matrix polynomials of smallest possible size (i.e. without degree canceling).*

Representing real zero polynomials as determinants of larger matrix polynomials corresponds to representing hyperbolic polynomials after multiplication with some power of x_0.

One can now formulate the Theorem of Helton and Vinnikov, as stated in Theorem 3.1.1, in the context of hyperbolic polynomials. It then turns out to be precisely the solution to the so-called *Lax Conjecture*, as observed by Lewis, Parrilo and Ramana [28].

Theorem 3.2.17 (Helton & Vinnikov)**.** *Let* $q = \mathbb{R}[x_0, x_1, x_2]$ *be a hyperbolic polynomial of degree* d*. Then there are* $M_1, M_2 \in \mathrm{Sym}_d(\mathbb{R})$ *with*

$$q = \det\left(x_0 I + x_1 M_1 + x_2 M_2\right).$$

3.3 Polynomials without Representations

The upper bounds on the size of matrix polynomials that we obtained in Section 3.2.1 can be used to show that many real zero polynomials do not admit a determinantal representation. This was an open question in the work of Helton and Vinnikov, first solved by Brändén. It turns out that in fact almost no real zero polynomial has a determinantal representation. We begin by explaining this fact. The results are from Netzer and Thom [41].

3.3.1 Almost no Polynomial has a Representation

Let $\mathbb{R}[\underline{x}]_d$ denote the finite dimensional subspace of polynomials of degree at most d. We again write $\mathcal{R}_{n,d}$ for the set of all real zero polynomials in

$\mathbb{R}[\underline{x}]_d$, and $\mathcal{D}_{n,d}$ for the set of polynomials in $\mathcal{R}_{n,d}$ having a determinantal representation (of any size). We have already seen in Proposition 2.2.9 that $\mathcal{R}_{n,d}$ is a closed and connected semi-algebraic set, having nonempty interior in the hyperplane of $\mathbb{R}[\underline{x}]_d$ defined by the condition $p(0) = 1$. We have seen in Proposition 2.5.4 that $\mathcal{R}_{n,d}$ is a regular set.

Lemma 3.3.1. *The set $\mathcal{R}_{n,d} \subseteq \mathbb{R}[\underline{x}]_d$ is of semi-algebraic dimension*

$$\binom{d+n}{d} - 1.$$

Proof. This is clear from the fact that $\mathcal{R}_{n,d}$ has nonempty interior in the hyperplane of $\mathbb{R}[\underline{x}]_d$ defined by the condition $p(0) = 1$. □

Theorem 3.3.2. *The set $\mathcal{D}_{n,d} \subseteq \mathbb{R}[\underline{x}]_d$ is a closed semi-algebraic set of dimension at most n^3d^2.*

Proof. Consider the semi-algebraic mapping

$$\begin{aligned} \det\colon \mathrm{H}_{nd}(\mathbb{C})^n &\to \mathbb{R}[\underline{x}]_{nd} \\ (M_1, \ldots, M_n) &\mapsto \det(I + x_1M_1 + \cdots + x_nM_n). \end{aligned}$$

The set $\mathcal{D}_{n,d}$ is the image of det intersected with $\mathbb{R}[\underline{x}]_d$, by Theorem 3.2.3. So $\mathcal{D}_{n,d}$ is semi-algebraic and of dimension at most

$$\dim_{\mathbb{R}} \mathrm{H}_{nd}(\mathbb{C})^n = n^3d^2.$$

Now let $(p_j)_{j\in\mathbb{N}}$ be a sequence of polynomials from $\mathcal{D}_{n,d}$, converging to some polynomial $p \in \mathcal{R}_{n,d}$. Let $M_1^{(j)}, \ldots, M_n^{(j)}$ be matrices of size nd from a determinantal representation of p_j. Since $\mathcal{R}(p)$ contains some ball around the origin, and the degree of all p_j is at most d, we can assume that each $\mathcal{R}(p_j)$ contains some fixed ball around the origin. In view of Lemma 2.1.8, this means that the eigenvalues and thus the norms of all $M_i^{(j)}$ are simultaneously bounded. So we can assume that each $M_i^{(j)}$ converges to some M_i. By continuity of det, this yields a determinantal representation of p. □

Comparing the dimensions of $\mathcal{R}_{n,d}$ and $\mathcal{D}_{n,d}$ we get the following Corollary:

Corollary 3.3.3. *For either $d \geq 4$ fixed and large enough values of n, or for $n \geq 3$ fixed and large enough values of d, the generic polynomial in $\mathcal{R}_{n,d}$ does not have a determinantal representation.*

Note that the result of Corollary 3.3.3 is non-constructive. In the next sections we construct explicit real zero polynomials without a determinantal representation.

3.3.2 Convex Cones Examples

Among all real zero polynomials, the ones with $\mathcal{R}(p)$ containing a full-dimensional cone are easiest to use as counterexamples.

Theorem 3.3.4. *Let $p \in \mathbb{R}[\underline{x}]$ be a real zero polynomial of degree d, defining a rigidly convex set $\mathcal{R}(p)$ that contains a full-dimensional cone, but not a full line. If $n > \binom{d+1}{2}$, then p does not have a symmetric determinantal representation. If $n > d^2$, then p does not have a hermitian determinantal representation.*

Proof. If p had a determinantal representation, then it would have one of size d, by Theorem 3.2.5. On the other hand, the matrices $M_1, \ldots, M_n$ occuring in such a representation would be linearly independent, by Lemma 3.2.9. Comparing with the real dimension of the space of symmetric and hermitian matrices, we get $n \leq \binom{d+1}{2}$ in the symmetric case and $n \leq d^2$ in the hermitian case. This contradicts the assumption. □

As explained in Remark 3.2.6, Brändén has considered real zero polynomials that arise as shifts of hyperbolic polynomials. He proved that almost none of them has a determinantal representation (but no explicit example can be derived from that result). Since the rigidly convex sets of such polynomials contain a full-dimensional cone, we can apply Theorem 3.3.4 to see that in fact *none* of them has a determinantal representation.

Example 3.3.5. Consider $p_n = 1 - x_1^2 - \cdots - x_n^2$. For $n \geq 3$ we find that

$$\hat{p}_n = (x_0 + 1)^2 - x_1^2 - \cdots - x_n^2$$

is not realizable as the determinant of a symmetric linear matrix polynomial. For $n \geq 4$ it is not realizable as a hermitian determinant. Note that for $n = 3$ we can realize it as the determinant of the hermitian matrix

$$\begin{pmatrix} 1 + x_0 + x_1 & x_2 + ix_3 \\ x_2 - ix_3 & 1 + x_0 - x_1 \end{pmatrix}.$$

Splitting this matrix into a symmetric and a skew-symmetric part, and building the symmetric block matrix of size 4 as explained in Lemma 3.2.14, we get a symmetric determinantal representation of $\hat{p}_3^2$. We will show below that for any quadratic RZ-polynomial, a high enough power has a determinantal representation.

Example 3.3.6. We can apply Theorem 3.3.4 also to polynomials that do not arise as a shifted hyperbolic polynomial. Consider for example the real zero polynomial

$$q_n = (x_1 + \sqrt{2})^2 - x_2^2 - \cdots - x_n^2 - 1,$$

whose zero set is a two-sheeted hyperboloid. For $n \geq 5$, it does not have a hermitian determinantal representation, for $n = 4$ no symmetric determinantal representation.

The above result applies to cases where the number of variables in high, compared to the degree of the polynomial. The following result applies to cases where the degree is high, compared to the number of variables.

Theorem 3.3.7. *Let $p \in \mathbb{R}[\underline{x}]$ be a real zero polynomial of degree d, such that $\mathcal{R}(p)$ does not contain a full line. Further suppose $d \not\equiv 0, 1, 7 \bmod 8$, and for each $a \in \mathbb{R}^n$, the polynomial p_a has only simple zeros (including the zeros at infinity). If $n \geq 3$, then the shifted homogenization $\hat{p}$ does not have a symmetric determinantal representation. For $n \geq 4$, it does not have a hermitian representation.*

Proof. If $\hat{p}$ has a determinantal representation, then by applying Theorem 3.2.5 and dehomogenizing we see that p has a representation of size d. Thus the real space V spanned by the n matrices occuring in such a representation contains only matrices with simple eigenvalues, by Lemma 2.1.8. The dimension of V is n, by Lemma 3.2.9. This contradicts the main result of Friedland, Robbin and Sylvester [12], saying that such spaces can have dimension at most 2 and 3, respectively (Theorem B in the symmetric case, and Theorem D in the hermitian case). □

Remark 3.3.8. The work of Friedland, Robbin and Sylvester also contains results in the case that $d \equiv 0, 1$ or $7 \bmod 8$, which are more technical. Although they can be used to obtain results in the spirit of Theorem 3.3.7, we decided not to include them, to keep the exposition more concise.

Example 3.3.9. Consider again $p_n = 1 - x_1^2 - \cdots - x_n^2$. Theorem 3.3.7 is another way to see that for $n \geq 3$, $\hat{p}_n$ does not have a symmetric representation, and no hermitian one for $n \geq 4$. But we can now rise the degree by for example considering

$$p_{n,m} := p_n(1+p_n)(2+p_n)\cdots(m-1+p_n).$$

If $n \geq 3$ and m is not a multiple of 4, then the shifted homogenization $\hat{p}_{n,m}$ does not have a symmetric determinantal representation. For $n \geq 4$ the same is true with hermitian representations. This contrasts the fact that taking high enough *powers* of p_n results in a polynomial whose shifted homogenization has a representation, as we will show in Section 3.4.2.

So far, in all counterexamples $\mathcal{R}(p)$ contains a full-dimensional cone. We can also construct counterexamples with $\mathcal{R}(p)$ compact, using Theorem 3.3.2.

3.3.3 Compact Examples

Let again $\hat{p}$ be the shifted homogenization of a real zero polynomial p. Then $\hat{p}$ is again a real zero polynomial, and there are explicit such examples without a determinantal representation, as we have just shown.

We now multiply $\hat{p}$ with a real zero polynomial defining a ball of radius $\sqrt{r} > 1$ around the point $(-1, 0, \ldots, 0)$:

$$q_r = \hat{p} \cdot \frac{r}{r-1}\left(1 - \frac{1}{r}\left((x_0+1)^2 + x_1^2 + \cdots + x_n^2\right)\right).$$

Then $\mathcal{R}(q_r)$ is clearly compact. Now if q_r has a determinantal representation for *some* $r > 1$, it has a representation for *all* $r > 1$. This follows easily from the fact that q_r and q_s can be transformed to each other by shifting and scaling.

Now for $r \to \infty$, the polynomials q_r converge to $\hat{p}$, and in view of the closedness result from Theorem 3.3.2, none of the q_r can thus have a determinantal representation. Note that if no power of $\hat{p}$ has a determinantal representation, then no power of no q_r can have a determinantal representation, by the same argument.

Example 3.3.10. Take $\hat{p}_4 = (x_0+1)^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2$. We find that

$$2\left((x_0+1)^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2\right)\left(1 - \frac{1}{2}\left((x_0+1)^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2\right)\right)$$

does not have a determinantal representation.

Example 3.3.11. Let $\hat{p} \in \mathcal{R}_{8,4}$ be Brändén's example, constructed from the Vámos cube (see also Example 3.3.14 below). It is a real zero polynomial of which no power has a determinantal representation. As above, by multiplying with a suitably shifted ball, we get a polynomial q with $\mathcal{R}(q)$ compact, and no power of q has a determinantal representation.

Remark 3.3.12. Note that if we multiply *any* $p \in \mathcal{R}_{n,d} \setminus \mathcal{D}_{n,d}$ with

$$1 - \frac{1}{r}\left(x_1^2 + x_1^2 + \cdots + x_n^2\right),$$

then for some large enough value of r, the result will be a polynomial without a determinantal representation, defining a compact set.

3.3.4 Renegar Derivatives

Since the Renegar derivative $\partial_R(p)$ of a real zero polynomial is less complex than p, one could conjecture that $\partial_R(p)$ admits a determinantal representation, if p does. Sanyal [52] has shown that this is true if p is a product of linear polynomials (so the corresponding spectrahedron is a polyhedron). In fact, if $p = \ell_1 \cdots \ell_d$ with $\ell_i(0) = 1$, then

$$\partial_R(p) = \sum_{i=1}^{d} \prod_{i \neq j} \ell_j = \det \begin{pmatrix} \ell_1 + \ell_d & \ell_d & \cdots & \ell_d \\ \ell_d & \ell_2 + \ell_d & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ell_d \\ \ell_d & \cdots & \ell_d & \ell_{d-1} + \ell_d \end{pmatrix},$$

and this representation can again be reduced to a monic one. Sanyal has also shown that the same result does not remain valid for higher Renegar derivatives. He considered hyperbolic polynomials only. As we have explained in Section 3.2.3, the representation of real zero polynomials is slightly less restrictive than the representation of hyperbolic polynomials. So we give an additional example to show that in fact the Renegar derivative of a representable polynomial is not necessarily representable, even in the case of real zero polynomials.

Example 3.3.13. Consider the real zero polynomial

$$p := \hat{p}_4 = (x_0 + 1)^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2.$$

In Section 3.4.2 it will be shown that p^2 admits a hermitian determinantal representation of size 4, which is the smallest possible size. We now claim that $\partial_R(p^2)$ does not have a determinantal representation. Indeed we have

$$\partial_R(p^2) = 2p \cdot \partial_R(p) = 4p \cdot (x_0 + 1).$$

If this polynomial had a determinantal representation, then it would have one of size 3, by Theorem 3.2.5. Evaluating at $x_0 = 0$ yields

$$q = 1 - x_1^2 - x_2^2 - x_3^3 - x_4^2 = \det(I + x_1 M_1 + x_2 M_2 + x_3 M_3 + x_4 M_4),$$

for some $M_i \in \mathrm{H}_3(\mathbb{C})$. By Lemma 2.1.8, each nonzero linear combination of $M_1, \ldots, M_4$ has only simple eigenvalues. Indeed there are always two nonzero eigenvalues coming from the zeros of q, and the zero eigenvalue, since the matrices are of size 3. But by Theorem D in Friedland, Robbin and Sylvester [12], such a space of matrices is of dimension at most 3. This contradiction shows that $\partial_R(p^2)$ does not admit a hermitian determinantal representation.

3.3.5 Polynomials of which no Power has a Representation

In this section we turn to the question whether some power of a real zero polynomial has a determinantal representation. This would be enough for the desired classification of spectrahedra, since $\mathcal{R}(p) = \mathcal{R}(p^r)$ holds for all $r \geq 1$. However, Brändén has provided an example in [8], constructed from the Vámos cube, which shows that this is not always possible. We roughly explain his idea.

Example 3.3.14 (Brändén)**.** Consider the cube in Figure 3.1, the so-called *Vámos cube.* Its *set of bases* $\mathcal{B}$ consists of all four element subsets of $\{1, \ldots, 8\}$ that do not lie in an affine hyperplane. Now define

$$q := \sum_{B \in \mathcal{B}} \prod_{i \in B} x_i,$$

a degree four polynomial in $\mathbb{R}[x_1, \ldots, x_8]$. It contains as its terms the product of any choice of four pairwisely different variables, except for the following five:

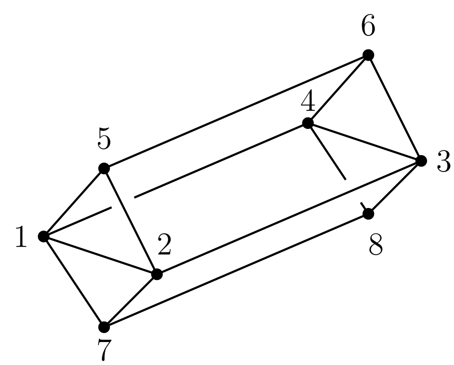
$$x_1x_4x_5x_6, x_2x_3x_5x_6, x_2x_3x_7x_8, x_1x_4x_7x_8, x_1x_2x_3x_4.$$

It was proven by Wagner and Wei [60] that q is hyperbolic (not exactly as in our definition; they use the direction $(1, \dots, 1)$ instead of $(1, 0, \dots, 0)$). Then it is easy to see that

$$p = q(x_1 + 1, \dots, x_8 + 1)$$

is a real zero polynomial. If any power of p had a determinantal representation, then a power of q would have a homogeneous determinantal representation. From the occuring matrices one could construct a subspace arrangement that realizes a multiple of the matroid defined by the Vámos cube. Such realized polymatroids fulfill the *Ingleton inequalities*. Then the Vámos matroid itself would fulfill these inequalities, which is known not to be the case.

Figure 3.1:



We will now give a different criterion, taken from Netzer, Plaumann and Thom [39], to show that no power of a real zero polynomial has a determinantal representation. It is based on sums of squares decompositions of the Hermite matrix $\mathcal{H}(p)$ from Section 2.2.

Lemma 3.3.15. *Let $p \in \mathbb{R}[\underline{x}]$ be a real zero polynomial of degree d. Assume that p^r has a determinantal representation of size k:*

$$p^r = \det\left(I + x_1 M_1 + \dots + x_n M_n\right).$$

Write $\underline{x} \circ M := x_1 M_1 + \dots + x_n M_n$. Then

$$r \cdot \mathcal{H}(p) + (k - rd) \cdot E_{11} = \left(\operatorname{tr}\left((-\underline{x} \circ M)^{i+j-2}\right)\right)_{i,j=1,\dots,d}.$$

Here E_{11} is the matrix with a one in the $(1,1)$-entry and zeros elsewhere.

Proof. For each $a \in \mathbb{R}^n$, the trace of $(-a \circ M)^s$ is the s-power sum of the (nonzero) eigenvalues of $-a \circ M$. These eigenvalues are the inverses of the zeros of p_a, by Lemma 2.1.8, but each such zero gives rise to r many eigenvalues. Since the zeros of p_a correspond to the inverses of the zeros of the opposite polynomial $\tilde{p}_a$, the trace of $(-a \circ M)^s$ equals the s-power sum of the zeros of $\tilde{p}_a$, multiplied with r. This proves the claim. Note that in the $(1,1)$-entry we need to add $k - rd$ on the left hand side of the equation, since the trace of the identity matrix of size k is k, and the $(1,1)$-entry of $\mathcal{H}(p)$ is d. □

Definition 3.3.16. For a matrix polynomial $H \in \mathrm{Sym}_k(\mathbb{R}[\underline{x}])$ we say that H is a *sum of squares (sos)*, if

$$H = Q^t Q$$

for some $l \in \mathbb{N}$ and $Q \in \mathrm{M}_{l \times k}(\mathbb{R}[\underline{x}])$.

Note that if H is sos, then H is clearly positive semidefinite at each point of $\mathbb{R}^n$. Note also that the converse is not true in general. Even in the case $k = 1$, there are well known examples in Real Algebra, of globally nonnegative polynomials that are not sums of squares of polynomials (see for example Prestel and Delzell [46] or Marshall [32]). Finally recall Proposition 2.2.6 from Section 2.2. It said that p is a real zero polynomial if and only if its Hermite matrix $\mathcal{H}(p)$ is positive semidefinite at each point of $\mathbb{R}^n$. We now find the following:

Theorem 3.3.17. *Let $p \in \mathbb{R}[\underline{x}]$ be a real zero polynomial of degree d. If p^r has a determinantal representation of size k, then*

$$\mathcal{H}(p) + (k/r - d) \cdot E_{11}$$

is a sum of squares. Again, E_{11} is the matrix with a one in the $(1,1)$-entry and zeros elsewhere. In particular, if $k = rd$, then $\mathcal{H}(p)$ is a sum of squares.

Proof. First note that in view of Lemma 3.2.14, we can assume that the determinantal representation is real symmetric. So let

$$p^r = \det(I + x_1 M_1 + \cdots + x_n M_n)$$

be such a representation of size k. Write $P = -\underline{x} \circ M$ and denote by P^m_{rs} the (r,s)-entry of P^m. Set

$$Q_{rs} = \left(P^0_{rs}, \ldots, P^{d-1}_{rs}\right)^t \in \mathbb{R}[\underline{x}]^d.$$

Then

$$\begin{aligned}\sum_{r,s=1}^{k} Q_{rs}Q_{rs}^t = \left(\sum_{r,s=1}^{k} P^{i-1}_{rs}P^{j-1}_{rs}\right)_{i,j=1,\ldots,d} &= \left(\operatorname{tr}(P^{i-1}P^{j-1})\right)_{i,j} \\ &= r \cdot \mathcal{H}(p) + (k - rd) \cdot E_{11},\end{aligned}$$

by Lemma 3.3.15. Writing the vectors Q_{rs} into the rows of a matrix Q and dividing by r proves the result. □

Remark 3.3.18. We have seen in Theorem 3.2.5 that if the rigidly convex set $\mathcal{R}(p)$ contains a full-dimensional cone, then p^r has a determinantal representation of size $k = rd$, if it has any at all. So $\mathcal{H}(p)$ is a sum of squares in that case. Brändén's counterexample does contain a full-dimensional cone, since it is a shifted hyperbolic polynomial.

Remark 3.3.19. By Theorem 3.2.3, p^r always has a determinantal representation of size $k = rdn$, if it has any at all. Thus $\mathcal{H}(p) + d(n-1) \cdot E_{11}$ is sos in that case. Since r cancels out here, one can show that *no power* has a determinantal representation, by showing that a *single* matrix polynomial is not sos.

Example 3.3.20. It is not hard to compute the Hermite matrix for Brändén's polynomial p. Unfortunately it is too complicated to check the sos condition by hand. If one uses a computer algebra system specialized on sums of squares, such as the matlab toolbox Yalmip, one obtains that $\mathcal{H}(p)$ is *not* a sum of squares. The same is true for a close enough smooth approximation of Brändén's polynomial, as described in Section 2.5 of Chapter 2. This is exactly what one expects, since the cone of sums of squares of matrices is closed. So both the Brändén polynomial and its close smooth approximations are examples of polynomials, of which no power has a determinantal representation.

Remark 3.3.21. Note that the sums of squares decomposition we get in Theorem 3.3.17 is very special. In the case $k = rd$ we indeed find

$$r \cdot \mathcal{H}(p) = \sum_{r,s=1}^{k} Q_{rs}Q_{rs}^t$$

for certain $Q_{rs} \in \mathbb{R}[\underline{x}]^d$ with $Q_{rs} = Q_{sr}$. If we write $Q_{rs,i}$ for the i-th entry of the vector Q_{rs}, and set

$$Q_i := \left(Q_{rs,i}\right)_{r,s},$$

the matrices Q_i fulfill the following relations:

$$Q_i = (Q_2)^{i-1} \text{ for } i = 1, \ldots, d$$

and

$$(Q_2)^d = -\sum_{i=1}^{d} p_{d-i+1} Q_i.$$

Now one can proof that with $\mathcal{M} := \left(Q_{rs,2}\right)_{r,s}$ we get back

$$\det(I - \mathcal{M}) = p^r.$$

This is essentially proven in Theorem 3.4.10 below. A general sum of squares decomposition of $\mathcal{H}(p)$ will however hardly be of that special form. How to still get back a determinantal representation from a sums of squares decomposition of the Hermite matrix will be the content of Section 3.4.1 below.

We will show in Theorem 3.4.15 below, that for quadratic real zero polynomials, a high enough power always admits a determinantal representation of minimal size. In view of Theorem 3.3.17, the Hermite matrix is always a sum of squares, for quadratic real zero polynomials. This can however be shown directly. We finish the section with this result.

Example 3.3.22. Let $p \in \mathbb{R}[\underline{x}]$ be a quadratic polynomial. Write

$$p = \underline{x}^t A \underline{x} + b^t \underline{x} + 1$$

with $A \in \operatorname{Sym}_n(\mathbb{R})$ and $b \in \mathbb{R}^n$. Then p is a real zero polynomial if and only if $bb^t - 4A \succeq 0$, as is easily checked. We find $\tilde{p} = \underline{x}^t A \underline{x} + b^t \underline{x} \cdot t + t^2$, and so we compute

$$\mathcal{H}(p) = \begin{pmatrix} 2 & -b^t \underline{x} \\ -b^t \underline{x} & \underline{x}^t (bb^t - 2A) \underline{x} \end{pmatrix}.$$

Write $bb^t - 4A = \sum_{i=1}^n v_i v_i^t$ as a sum of squares of vectors $v_i \in \mathbb{R}^n$. Set

$$Q = \begin{pmatrix} 1 & -\frac{1}{2} b^t \underline{x} \\ 0 & \frac{1}{2} v_1^t \underline{x} \\ \vdots & \vdots \\ 0 & \frac{1}{2} v_n^t \underline{x} \end{pmatrix}.$$

Then one immediately checks

$$\mathcal{H}(p) = 2 \cdot Q^t Q.$$

3.4 Polynomials with Representations

In this section we describe several positive results concerning determinantal representations of real zero polynomials. We start with a general construction method, that can be used to construct determinantal representations from sums of squares decompositions of the Hermite matrix. We then characterize real zero polynomials of which some power has a representation, in terms of an algebra having a finite dimensional representation. We finally show that each real zero polynomial admits a *rational* linear determinantal representation, i.e. a representation with denominators.

3.4.1 A General Construction Method

Let $p = 1 + p_1 + \cdots + p_d \in \mathbb{R}[\underline{x}]$ be a real zero polynomial of degree d. We again denote by $\mathcal{H}(p)$ the Hermite matrix of p. The following lemma is a slight adjustment of the main result of Gondard and Ribenboim [13] to our setup.

Lemma 3.4.1. *There is some nonzero homogeneous polynomial $q \in \mathbb{R}[\underline{x}]$ such that*

$$q^2 \cdot \mathcal{H}(p) = Q^t Q$$

for some $Q \in \mathrm{M}_{k\times d}\left(\mathbb{R}[\underline{x}]\right)$.

Proof. $\mathcal{H}(p)$ is positive semidefinite at each point of $\mathbb{R}^n$, so by Gondard and Ribenboim [13] there is some nonzero polynomial $q \in \mathbb{R}[\underline{x}]$ such that $q^2\mathcal{H}(p) = Q^tQ$ for some $Q \in \mathrm{M}_{k\times d}(\mathbb{R}[\underline{x}])$. We have to show that we can choose q to be homogeneous.

Write $q = q_r + q_{r+1} + \cdots + q_R$, where each q_i is homogeneous of degree i, and $p_r \neq 0, p_R \neq 0$. Since the i-th diagonal entry in $\mathcal{H}(p)$ is homogeneous of degree $2(i-1)$, each entry in the i-th column of Q has homogeneous parts only of degree between $r+i-1$ and $R+i-1$. Let $Q_{\min}$ be the matrix one obtains from Q by choosing only the homogeneous part of degree $r+i-1$ of each entry in each i-th column. Write $Q = Q_{\min} + \widetilde{Q}$ and note that all entries in the i-th column of $\widetilde{Q}$ start with homogeneous parts of degree at least $r+i$. We now have

$$q^2\mathcal{H}(p) = Q_{\min}^t Q_{\min} + Q_{\min}^t \widetilde{Q} + \widetilde{Q}^t Q_{\min} + \widetilde{Q}^t \widetilde{Q}.$$

By comparing degrees we find

$$q_r^2\mathcal{H}(p) = Q_{\min}^t Q_{\min},$$

the desired result. $\square$

We will fix a sums of squares representation with denominators as in Lemma 3.4.1 for the rest of this section. Write r for the degree of the homogeneous polynomial q. Again set

$$\tilde{p} = t^d \cdot p\left(\frac{\underline{x}}{t}\right) = t^d + p_1 t^{d-1} + \cdots + p_d \in \mathbb{R}[\underline{x}, t].$$

We consider the $\mathbb{R}[\underline{x}]$-module

$$A := \mathbb{R}[\underline{x}, t]/\tilde{p} = \bigoplus_{i=0}^{d-1} \mathbb{R}[\underline{x}] \cdot t^i = \mathbb{R}[\underline{x}]^d.$$

We can equip A with a symmetric $\mathbb{R}[\underline{x}]$-bilinear and $\mathbb{R}[\underline{x}]$-valued map $\langle\cdot,\cdot\rangle_p$. Indeed for$f = (f_1, \ldots, f_d)$ and $g = (g_1, \ldots, g_d)$ we define

$$\langle f, g\rangle_p := f^t\left(q^2\mathcal{H}(p)\right) g.$$

Since $\tilde{p}$ is homogeneous, the standard grading of $\mathbb{R}[\underline{x}, t]$ induces a grading on A. We shift it by r, so it fulfills

$$\deg(t^i) = r + i$$

for $i = 0, \ldots, d-1$. This makes $A = \bigoplus_{m\in\mathbb{N}} A_m$ a graded $\mathbb{R}[\underline{x}]$-module, where $\mathbb{R}[\underline{x}]$ is equipped with the standard grading. We next consider multiplication with t on A. This is a well defined and $\mathbb{R}[\underline{x}]$-linear map, given by the following formula:

$$\begin{aligned} \mathcal{L}_t \colon A &\to A \\ (f_1, \ldots, f_d) &\mapsto (-p_d f_d, f_1 - p_{d-1} f_d, \ldots, f_{d-1} - p_1 f_d). \end{aligned}$$

Note that $\mathcal{L}_t$ is of degree 1 with respect to the grading, i.e. if $f \in A_m$, then $\mathcal{L}_t(f) \in A_{m+1}$. Note also that the representing matrix of $\mathcal{L}_t$ with respect to the standard basis is

$$\mathcal{L}_t = \begin{pmatrix} 0 & 0 & 0 & -p_d \\ 1 & 0 & 0 & -p_{d-1} \\ 0 & \ddots & 0 & \vdots \\ 0 & \cdots & 1 & -p_1 \end{pmatrix},$$

the so called *companion matrix* of p. It is well known and easy to see that

$$\det(I - \mathcal{L}_t) = p.$$

Lemma 3.4.2. *$\mathcal{L}_t$ is self-adjoint with respect to $\langle \cdot, \cdot \rangle_p$, i.e.*

$$\langle \mathcal{L}_t f, g \rangle_p = \langle f, \mathcal{L}_t g \rangle_p$$

holds for all $f, g \in A$.

Proof. We can assume without loss of generality that $q = 1$. It is enough to show $\langle \mathcal{L}_t e_i, e_j \rangle_p = \langle e_i, \mathcal{L}_t e_j \rangle_p$ for all i, j, where e_i is the tuple with a one in the i-th component and zeros elsewhere. For $i, j < d$ this follows from the fact that $\mathcal{H}(p)$ is a Hankel matrix. For $i = j = d$ this is clear from symmetry. So assume $j < i = d$. We find

$$\langle \mathcal{L}_t e_d, e_j \rangle_p = -\sum_{i=1}^{d} p_{d-i+1} e_i \mathcal{H}(p) e_j = -\sum_{i=1}^{d} p_{d-i+1} N_{i+j-2},$$

where N_k is the k-th Newton sum of $\widetilde{p}$. On the other hand we compute

$$\langle e_d, \mathcal{L}_t e_j \rangle_p = \langle e_d, e_{j+1} \rangle_p = N_{d+j-1}.$$

So we have to show

$$0 = \sum_{i=0}^{d} p_{d-i} N_{i+j-1},$$

where $p_0 := 1$. This statement is equivalent to

$$0 = \sum_{i=0}^{d} p_i N_{k-i},$$

where $k := d + j - 1$. This last equation however follows immediately from the Newton identities:

$$kp_k + \sum_{i=0}^{k-1} p_i N_{k-i} = 0,$$

where $p_k := 0$ if $k > d$. □

We now consider the $\mathbb{R}[\underline{x}]$-linear mapping defined as multiplication by Q:

$$\begin{aligned} Q\colon A &\to \mathbb{R}[\underline{x}]^k \\ f &\mapsto Qf. \end{aligned}$$

If we equip the $\mathbb{R}[\underline{x}]$-module $B = \mathbb{R}[\underline{x}]^k$ with the canonical $\mathbb{R}[\underline{x}]$-valued bilinearform $\langle \cdot, \cdot \rangle$, then Q is an isometry, in the following sense:

$$\langle f, g \rangle_p = \langle Qf, Qg \rangle.$$

Lemma 3.4.3. *If p is square-free, then Q is injective.*

Proof. If $Qf = 0$, then

$$0 = \langle Qf, Qf \rangle = \langle f, f \rangle_p = q^2 \cdot f^t \mathcal{H}(p) f.$$

For each a for which p_a has only distinct roots, the matrix $\mathcal{H}(p)(a)$ is positive definite. So $f(a) = 0$ for generic a, and thus $f = 0$. □

From the degree structure of $\mathcal{H}(p)$ we see that each entry in the i-th column of Q is homogeneous of degree $r + i - 1$. So Q is of degree 0, if we equip B with the canonical grading. All in all we have the following diagram of $\mathbb{R}[\underline{x}]$-linear maps:

$$\begin{tikzcd} \mathbb{R}[\underline{x}]^d = A \arrow[r, "Q"] \arrow[d, "\mathcal{L}_t"] & B = \mathbb{R}[\underline{x}]^k \\ \mathbb{R}[\underline{x}]^d = A \arrow[r, "Q"] & B = \mathbb{R}[\underline{x}]^k \end{tikzcd}$$

We repeat the most important facts:

- $\mathcal{L}_t$ is self-adjoint with respect to $\langle \cdot, \cdot \rangle_p$ and of degree 1 with respect to the introduced grading on A.
- $\det(I - \mathcal{L}_t) = p$.

- Q is an isometry, and of degree 0.

The following is our main result:

Theorem 3.4.4. *Let p be a square-free real zero polynomial. Assume there is a homogeneous symmetric linear matrix polynomial $\mathcal{M}$ of size k, such that multiplication with M, as a mapping from B to B, makes this diagram commute:*

$$\begin{array}{ccc} \mathbb{R}[\underline{x}]^d = A & \xrightarrow{Q} & B = \mathbb{R}[\underline{x}]^k \\ \downarrow \mathcal{L}_t & & \downarrow \mathcal{M} \\ \mathbb{R}[\underline{x}]^d = A & \xrightarrow{Q} & B = \mathbb{R}[\underline{x}]^k \end{array}$$

Then $\det(I - \mathcal{M})$ *contains p as a factor.*

Proof. For generic $a \in \mathbb{R}^n$, the mapping $Q(a)$ is injective. So all eigenvalues of $\mathcal{L}_t(a)$ are also eigenvalues of $\mathcal{M}(a)$. The eigenvalues of $\mathcal{L}_t(a)$ are precisely the zeros of $\hat{p}_a$, i.e. the inverses of the zeros of p_a. So $q = \det(I - \mathcal{M})$ vanishes on the zero set of p, by Proposition 2.1.8. Since p is square-free, this implies the claim, using the fact that irreducible real zero polynomials define real radical ideals, see Bochnak, Coste and Roy [7], Theorem 4.5.1(v). □

Remark 3.4.5. Note that whether there is such $\mathcal{M}$ can be decided by solving a system of linear equations. Indeed set $\mathcal{M} = x_1 M_1 + \cdots + x_n M_n$, where the M_i are symmetric matrices with indeterminate entries. The matrix equation $\mathcal{M}Q = Q\mathcal{L}_t$ can be considered entrywise, and comparison of the coefficients of the occuring polynomials in $\underline{x}$ gives rise to a system of linear equations in the entries of the M_i.

Example 3.4.6. Let $p \in \mathbb{R}[\underline{x}]$ be quadratic. Write $p = \underline{x}^t A \underline{x} + b^t \underline{x} + 1$ with $A \in \mathrm{Sym}_n(\mathbb{R})$ and $b \in \mathbb{R}^n$. We have seen in Example 3.3.22 that $\mathcal{H}(p)$ admits a sums of squares decomposition if p is real zero, given by the following matrix Q:

$$Q = \sqrt{2} \cdot \begin{pmatrix} 1 & -\frac{1}{2} b^t \underline{x} \\ 0 & \frac{1}{2} v_1^t \underline{x} \\ \vdots & \vdots \\ 0 & \frac{1}{2} v_n^t \underline{x} \end{pmatrix},$$

if $bb^t - 4A = \sum_{i=1}^n v_i v_i^t$. One now easily checks that there is some homogeneous linear matrix polynomial $\mathcal{M}$ that makes the diagram commute. One takes

$$\mathcal{M} = \frac{1}{2} \cdot \begin{pmatrix} -b^t\underline{x} & v_1^t\underline{x} & \cdots & v_n^t\underline{x} \\ v_1^t\underline{x} & -b^t\underline{x} & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ v_n^t\underline{x} & 0 & 0 & -b^t\underline{x} \end{pmatrix}.$$

One also checks easily that

$$\det (I - \mathcal{M}) = (1 + \frac{1}{2} \cdot b^t\underline{x})^{n-1} \cdot p$$

holds.

As an explicit example consider $p = (x_1 + \sqrt{2})^2 - x_2^2 - x_3^2 - x_4^2 - x_5^2$, which does not admit a determinantal representation. The described procedure now gives rise to the following linear matrix polynomial:

$$\mathcal{M} = \begin{pmatrix} -\sqrt{2}x_1 & x_1 & x_2 & x_3 & x_4 & x_5 \\ x_1 & -\sqrt{2}x_1 & 0 & 0 & 0 & 0 \\ x_2 & 0 & -\sqrt{2}x_1 & 0 & 0 & 0 \\ x_3 & 0 & 0 & -\sqrt{2}x_1 & 0 & 0 \\ x_4 & 0 & 0 & 0 & -\sqrt{2}x_1 & 0 \\ x_5 & 0 & 0 & 0 & 0 & -\sqrt{2}x_1 \end{pmatrix}$$

and finally

$$\det(I - \mathcal{M}) = (1 + \sqrt{2}x_1)^4 \cdot p.$$

Example 3.4.7. There are also examples where there is no such $\mathcal{M}$ that makes the diagram commute. Consider the cubic $p = (x_1 - 1)^2(x_1 + 1) - x_2^2$. One computes

$$H = \begin{pmatrix} 3 & x_1 & 3x_1^2 + 2x_2^2 \\ x_1 & 3x_1^2 + 2x_2^2 & x_1^3 + 3x_1x_2^2 \\ 3x_1^2 + 2x_2^2 & x_1^3 + 3x_1x_2^2 & 3x_1^4 + 8x_1^2x_2^2 + 2x_2^4 \end{pmatrix} = Q^tQ,$$

where

$$Q = \begin{pmatrix} 0 & x_2 & a \cdot x_1x_2 \\ 0 & -x_2 & b \cdot x_1x_2 \\ \sqrt{2} & \sqrt{2}x_1 & \sqrt{2}(x_1^2 + x_2^2) \\ 1 & -x_1 & x_1^2 \end{pmatrix}$$

and $a = \frac{1}{2}(\sqrt{7}+1)$, $b = \frac{1}{2}(\sqrt{7}-1)$. The equation $\mathcal{M}Q = Q\mathcal{L}_t$ has 12 entries, each one giving rise to several linear equations, by comparing coefficients of polynomials in $\underline{x}$. Already the equations obtained by the first two rows of the equation $\mathcal{M}Q = Q\mathcal{L}_t$ are checked to be unsolvable. We want to thank Rainer Sinn and Cynthia Vinzant for helping us find this example.

3.4.2 Representing Powers of Real Zero Polynomials

In this section we characterize polynomials for which some power admits a determinantal representation. The approach is as follows. For a given real zero polynomial p, one constructs a (non-commutative) algebra with involution, called *the generalized Clifford algebra associated with* p. We show that representing some power of p as a determinant is the same as finding a finite dimensional $*$-representation of the algebra. In the case of quadratic polynomials we can explicitly construct such an algebra representation. This leads to explicit determinantal representations of powers of quadratic polynomials.

A similar approach has been used for the problem of linearizing forms, by Heerema [17], Roby [50] and Childs [10], among others. A solution to their problem implies a determinantal representation for the polynomial, but not necessarily a hermitian one, and also without the matrix M_0 being positive semidefinite. Further, Pappacena [43] has used an algebra as below to realize polynomials as minimal polynomials of matrix pencils. From this one can also deduce determinantal representations, this time even monic, i.e. with $M_0 = I$, but still not necessarily with all other matrices hermitian. We will see that the strive for hermitian representations needs some more work in general.

A Generalized Clifford Algebra

Let $p \in \mathbb{R}[\underline{x}]$ be a real zero polynomial of degree $d \geq 1$, and

$$\widetilde{p}(x_0, \ldots, x_n) := x_0^d \cdot p\left(\frac{\underline{x}}{x_0}\right)$$

its homogenization. In the free non-commutative algebra

$$\mathbb{C}\langle \underline{z} \rangle = \mathbb{C}\langle z_1, \ldots, z_n \rangle$$

consider the polynomial

$$h_a := \widetilde{p}(-a_1 z_1 - \cdots - a_n z_n, a_1, \ldots, a_n),$$

for $a \in \mathbb{R}^n$. Note that if $p = 1 + p_1 + \cdots + p_d$ is the decomposition of p into its homogeneous parts, and if we abbreviate $a_1z_1 + \cdots + a_nz_n$ by $a \circ \underline{z}$, then

$$h_a = (-1)^d(a \circ \underline{z})^d + (-1)^{d-1}(a \circ \underline{z})^{d-1}p_1(a) + \cdots - (a \circ \underline{z})p_{d-1}(a) + p_d(a).$$

Now let $J(p)$ be the two-sided ideal in $\mathbb{C}\langle\underline{z}\rangle$ generated by all the polynomials h_a, with $a \in \mathbb{R}^n$. We equip $\mathbb{C}\langle\underline{z}\rangle$ with the involution defined by $z_j^* = z_j$, for all j. Then $J(p)$ is a $*$-ideal and we can define the involution on the quotient.

Definition 3.4.8. We call the $*$-algebra

$$\mathcal{A}(p) := \mathbb{C}\langle z_1, \ldots, z_n\rangle / J(p)$$

the *generalized Clifford algebra associated with* p*.*

Remark 3.4.9. Note that the ideal $J(p)$ is finitely generated, although we used infinitely many generators to define it. Write

$$h_a = \sum_{\alpha \in \mathbb{N}^n, |\alpha| = d} a^\alpha q_\alpha$$

for suitable $q_\alpha \in \mathbb{C}\langle\underline{z}\rangle$. It is then easy so check that the q_α generate the ideal $J(p)$.

Under a *finite dimensional unital $*$-representation* of $\mathcal{A}(p)$ we will in the following understand an algebra homomorphism $\mathcal{A}(p) \to \mathrm{M}_k(\mathbb{C})$ for some $k \in \mathbb{N}$, preserving the unit and the involution. We call k the *dimension* of the representation. The following is our main result in this section.

Theorem 3.4.10. *Let $p \in \mathbb{R}[\underline{x}]$ be a real zero polynomial of degree $d \geq 1$.*

(1) If some power p^r has a determinantal representation of size rd, then $\mathcal{A}(p)$ admits a unital $*$-representation of dimension rd.

(2) If p is irreducible and $\mathcal{A}(p)$ admits a unital $*$-representation of dimension k, then $k = rd$ for some $r \in \mathbb{N}$, and p^r has a determinantal representation of size rd.

Proof. For (1) assume $p^r = \det(I + x_1M_1 + \cdots + x_nM_n)$ for some hermitian matrices M_j of size rd. Consider the unital $*$-algebra homomorphism

$$\varphi\colon \mathbb{C}\langle z_1,\ldots,z_n\rangle \to \mathrm{M}_{rd}(\mathbb{C}), z_i \mapsto M_i.$$

For any $a \in \mathbb{R}^n$ we know by Lemma 2.1.8 that the eigenvalues of $a_1M_1 + \cdots + a_nM_n$ arise from the zeros of p_a^r by the rule $\lambda \mapsto -\frac{1}{\lambda}$, including possible zeros at infinity. These eigenvalues are precisely the zeros of the univariate polynomial $\widetilde{p}(-t, a_1,\ldots,a_n)$, so the minimal polynomial of $a_1M_1+\cdots+a_nM_n$ divides $\widetilde{p}(-t, a_1,\ldots,a_n)$. This means $\varphi(h_a) = 0$, so φ induces a representation of $\mathcal{A}(p)$ as desired.

For (2) let $\varphi\colon \mathcal{A}(p) \to \mathrm{M}_k(\mathbb{C})$ be a unital $*$-algebra homomorphism. Set $M_i := \varphi\left(z_i + J(p)\right)$, consider the linear matrix polynomial

$$\mathcal{M} = I + x_1M_1 + \cdots + x_nM_n$$

and its determinant $q = \det\mathcal{M}$. From the defining relations of $\mathcal{A}(p)$ we know

$$\widetilde{p}(-a_1M_1 - \cdots - a_nM_n, a_1,\ldots,a_n) = 0$$

for all $a \in \mathbb{R}^n$. So the eigenvalues of $a_1M_1 + \cdots + a_nM_n$ are among the $-\frac{1}{\lambda}$, where λ runs through the zeros of p_a (including possibly $\lambda = \infty$). Lemma 2.1.8 implies that the zeros of q are contained in the zeros of p, and also $\deg(q) = k$. Since every irreducible real zero polynomial defines a real radical ideal (which follows for example from Bochnak, Coste and Roy [7] Theorem 4.5.1(v)), the real Nullstellensatz implies that each irreducible factor of q divides p. So q divides some power of p, and since p is itself irreducible, $q = p^r$ for some $r \geq 1$. This now finally implies $k = rd$. $\square$

Remark 3.4.11. One could of course also define the generalized Clifford algebra as a quotient of the free algebra over the real numbers, instead of the complex numbers, as we did here. This would allow to characterize symmetric representations of powers of p. But in view of Lemma 3.2.14, that would only make sense when one is interested in determining the *lowest* possible power for which there exists a symmetric representation. Since the classification of algebras is often simpler over the complex numbers, we decided not to take this approach.

Example 3.4.12. Consider $p_n = 1 - x_1^2 - \cdots - x_n^2$. We find $\mathcal{A}(p_n)$ defined via the relations

$$(a_1 z_1 + \cdots + a_n z_n)^2 = \|a\|^2,$$

which is the classical Clifford Algebra $\mathrm{Cl}_n(\mathbb{C})$. We can also compute the finitely many relations described in Remark 3.4.9; they are

$$z_i^2 = 1 \text{ for } i = 1, \ldots, n$$

$$z_i z_j = -z_j z_i \text{ for } i \neq j.$$

It is well known that

$$\mathrm{Cl}_n(\mathbb{C}) \cong \mathrm{M}_k(\mathbb{C})$$

for even n and $k = 2^{\frac{n}{2}}$, and

$$\mathrm{Cl}_n(\mathbb{C}) \cong \mathrm{M}_k(\mathbb{C}) \oplus \mathrm{M}_k(\mathbb{C})$$

for n odd and $k = 2^{\frac{n-1}{2}}$. So $\mathrm{Cl}_n(\mathbb{C})$ admits a $*$-algebra homomorphism to $\mathrm{M}_k(\mathbb{C})$ with $k = 2^{\lfloor \frac{n}{2} \rfloor}$, for any n. Thus the $2^{\lfloor \frac{n}{2} \rfloor - 1}$-th power of p_n has a determinantal representation of size $2^{\lfloor \frac{n}{2} \rfloor}$. In the case of $n = 2m$ we can use the Brauer-Weyl matrices to generate the Clifford Algebra, as described in Brauer and Weyl [9]. Let

$$1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 1' := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, P := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Q := \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

Then consider the hermitian matrices

$$1' \otimes \cdots \otimes 1' \otimes P \otimes 1 \cdots \otimes 1$$

and

$$1' \otimes \cdots \otimes 1' \otimes Q \otimes 1 \cdots \otimes 1,$$

where $\otimes$ denotes the Kronecker (tensor) product of matrices, the product is of length m, and both P and Q run through all m possible positions in this product. The arising $2m = n$ matrices are hermitian and yield a determinantal representation of the 2^{m-1}-th power of p_n. In the case of n odd one uses the additional matrix $1' \otimes \cdots \otimes 1'$ to construct a representation of $\mathrm{Cl}_n(\mathbb{C})$. This yields for example

$$p_5^2 = \det \begin{pmatrix} 1+x_5 & x_1+ix_3 & x_2+ix_4 & 0 \\ x_1-ix_3 & 1-x_5 & 0 & -x_2-ix_4 \\ x_2-ix_4 & 0 & 1-x_5 & x_1+ix_3 \\ 0 & -x_2+ix_4 & x_1-ix_3 & 1+x_5 \end{pmatrix}$$

Note that in the odd case there is another representation of $\mathrm{Cl}_n(\mathbb{C})$, given by the respective negative matrices, which is not unitarily equivalent to the first one (in contrast to the even case, where these representations are equivalent).

The Quadratic Case

In this section we construct a finite dimensional $*$-representation of $\mathcal{A}(p)$, if p is quadratic. Note that already Pappacena [43] has proven $\mathcal{A}(p)$ to be isomorphic to the Clifford algebra in the quadratic case. We need to be more subtle, since we are looking for homomorphisms respecting the involution. We start with a lemma that was also noted by Pappacena, and include the proof for completeness.

Lemma 3.4.13. *If $p \in \mathbb{R}[\underline{x}]$ is a quadratic real zero polynomial, then*

$$\dim_{\mathbb{C}} \mathcal{A}(p) \leq 2^n.$$

Proof. Let V be the real subspace in $\mathcal{A}(p)$ spanned by the elements $z_i + J(p)$ and 1. Each element $v \in V$ fulfills a real quadratic relation $v^2 = rv + s$. For $v, w \in \mathcal{A}(p)$ write $v \equiv w$ if $v - w \in V$. Clearly $v^2 \equiv 0$ for all $v \in V$. We compute

$$0 \equiv ((z_i + J(p)) + (z_j + J(p)))^2 \equiv (z_i z_j + J(p)) + (z_j z_i + J(p)),$$

so $z_i z_j + J(p) \equiv -z_j z_i + J(p)$ holds in $\mathcal{A}(p)$. This proves that the elements

$$z_{i_1} \cdots z_{i_r} + J(p)$$

with $i_1 < \cdots < i_r$ generate $\mathcal{A}(p)$ as a vector space, which finishes the proof. $\square$

Write a quadratic real zero polynomial p as

$$p(\underline{x}) = \underline{x}^t A \underline{x} + b^t \underline{x} + 1$$

with $A \in \mathrm{Sym}_n(\mathbb{R})$ and $b \in \mathbb{R}^n$. Then $p_a(t) = a^tAa \cdot t^2 + b^ta \cdot t + 1$, and the condition that p_a has only real roots is $\frac{1}{4}a^tbb^ta - a^tAa \geq 0$. So p being a real zero polynomial is equivalent to

$$\frac{1}{4}bb^t - A \succeq 0,$$

and this matrix then has a positive semidefinite symmetric square root.

When we use the Clifford Algebra $\mathrm{Cl}_n(\mathbb{C})$ in the following, we denote its standard generators by $\sigma_1, \ldots, \sigma_n$. They fulfill the relations

$$\sigma_i^2 = 1, \sigma_i^* = \sigma_i \text{ and } \sigma_i\sigma_j = -\sigma_j\sigma_i \text{ for } i \neq j.$$

We denote by σ the column vector

$$\sigma = (\sigma_1, \ldots, \sigma_n)^t.$$

Proposition 3.4.14. *Let $p = \underline{x}^tA\underline{x} + b^t\underline{x} + 1 \in \mathbb{R}[x_1, \ldots, x_n]$ be a quadratic real zero polynomial. Then there is a unital $*$-algebra homomorphism*

$$\mathcal{A}(p) \to \mathrm{Cl}_n(\mathbb{C}),$$

defined by the rule

$$a_1z_1 + \cdots + a_nz_n + J(p) \mapsto \sigma^t\left(\frac{1}{4}bb^t - A\right)^{\frac{1}{2}}a + \frac{1}{2}b^ta$$

for all $a \in \mathbb{R}^n$. If $\frac{1}{4}bb^t - A$ is invertible, this is an isomorphism.

Proof. We abbreviate $\left(\frac{1}{4}bb^t - A\right)^{\frac{1}{2}}$ by C and $\sigma^tCa + \frac{1}{2}b^ta$ by c_a. We denote the entries of the real symmetric matrix Caa^tC by q_{ij} and compute in $\mathrm{Cl}_n(\mathbb{C})$:

$$\begin{aligned}
c_a^2 &= \sigma^tCaa^tC\sigma + b^ta\sigma^tCa + \frac{1}{4}(b^ta)^2 \\
&= \sum_{i,j}\sigma_iq_{ij}\sigma_j + b^ta\sigma^tCa + \frac{1}{4}(b^ta)^2 \\
&= \sum_i q_{ii} + \sum_{i<j}(\underbrace{q_{ij} - q_{ji}}_{=0})\sigma_i\sigma_j + b^ta\sigma^tCa + \frac{1}{4}(b^ta)^2 \\
&= \mathrm{tr}(Caa^tC) + b^ta\sigma^tCa + \frac{1}{4}(b^ta)^2 \\
&= \mathrm{tr}(a^tC^2a) + b^ta\sigma^tCa + \frac{1}{4}(b^ta)^2
\end{aligned}$$

$$
\begin{aligned}
&= a^t\left(\frac{1}{4}bb^t - A\right)a + b^t a\sigma^t Ca + \frac{1}{4}(b^t a)^2\\
&= \frac{1}{2}(b^t a)^2 + b^t a\sigma^t Ca - a^t Aa\\
&= b^t a \cdot c_a - a^t Aa.
\end{aligned}
$$

Now we define a unital $*$-algebra homomorphism

$$\varphi\colon \mathbb{C}\langle z_1,\dots,z_n\rangle \to \mathrm{Cl}_n(\mathbb{C}); \quad a_1z_1+\cdots+a_nz_n \mapsto c_a.$$

The ideal $J(p)$ is in our case generated by the polynomials

$$h_a = (a_1z_1+\cdots+a_nz_n)^2 - b^t a\cdot(a_1z_1+\cdots+a_nz_n) + a^t Aa,$$

so

$$\varphi(h_a) = c_a^2 - b^t a\cdot c_a + a^t Aa = 0.$$

Thus φ is well defined on $\mathcal{A}(p)$. In case that $\frac{1}{4}bb^t - A$ is invertible, φ is onto. So Lemma 3.4.13 finishes the proof, using that the vector space dimension of $\mathrm{Cl}_n(\mathbb{C})$ is 2^n. □

Now we can prove the main result from this section.

Theorem 3.4.15. *Let $p\in\mathbb{R}[x_1,\dots,x_n]$ be a quadratic real zero polynomial. Then for $r = 2^{\lfloor\frac{n}{2}\rfloor-1}$, p^r has a hermitian determinantal representation of size $2^{\lfloor\frac{n}{2}\rfloor}$.*

Proof. We have seen in Proposition 3.4.14 that there is a unital $*$-algebra homomorphism from $\mathcal{A}(p)$ to $\mathrm{Cl}_n(\mathbb{C})$. But as already described in Example 3.4.12, $\mathrm{Cl}_n(\mathbb{C})$ admits a unital $*$-algebra homomorphism to $\mathrm{M}_k(\mathbb{C})$, with $k = 2^{\lfloor\frac{n}{2}\rfloor}$. So we can apply Theorem 3.4.10 to finish the proof, noting that the case where p is reducible is trivial. □

As explained in the beginning of Chapter 3, this immediately implies the following:

Corollary 3.4.16. *Let $p\in\mathbb{R}[\underline{x}]$ be a quadratic real zero polynomial. Then $\mathcal{R}(p)$ is a spectrahedron.*

Remark 3.4.17. We can compute the determinantal representations in the setup of Theorem 3.4.15 explicitly. Proposition 3.4.14 gives an explicit morphism from $\mathcal{A}(p)$ to $\mathrm{Cl}_n(\mathbb{C})$, and this yields an explicit representation in $\mathrm{M}_k(\mathbb{C})$, using for example the Brauer-Weyl matrices (see Examples 3.4.19 and 3.4.20 below).

Also of interest is the question how many different representations for a real zero polynomial exist. Helton, Klep and McCullough [18] have for example characterized equivalent representations in terms of matricial spectrahedra, i.e. spectrahedra defined in $(\mathrm{Sym}_k(\mathbb{R}))^n$, instead of $\mathbb{R}^n$ only. Under a regularity condition on the polynomial, we see that the representations in Theorem 3.4.15 can be described completely, up to unitary equivalence.

Theorem 3.4.18. *Let $p = \underline{x}^t A \underline{x} + b^t \underline{x} + 1 \in \mathbb{R}[x_1, \ldots, x_n]$ be a quadratic real zero polynomial for which $\frac{1}{4}bb^t - A$ is invertible. Set $k = 2^{\lfloor \frac{n}{2} \rfloor}$.*

If p^r has a determinantal representation of size $2r$, for some $r \geq 1$, then r is a positive multiple of $\frac{k}{2}$. After a unitary change of variables, the representation splits into blocks of size k, each one representing $p^{\frac{k}{2}}$.

If n is even, then any two determinantal representations of $p^{\frac{k}{2}}$ of size k are unitarily equivalent. If n is odd then there are precisely two such representations, up to unitary equivalence.

Proof. Note that the regularity condition implies that p is irreducible. Now let first n be even. From Proposition 3.4.14 we know

$$\mathcal{A}(p) \cong \mathrm{Cl}_n(\mathbb{C}) \cong \mathrm{M}_k(\mathbb{C}).$$

A determinantal representation of p^r of size $2r$ gives rise to a $*$-algebra representation of $\mathrm{M}_k(\mathbb{C})$ of dimension $2r$. From the classification of $*$-subalgebras of matrix algebras we see that this representation splits into blocks, which are of size k since $\mathrm{M}_k(\mathbb{C})$ is simple. Finally, since every $*$-automorphism of a matrix algebra is conjugation with a unitary matrix, any two representations of size k are unitarily equivalent.

Now let n be odd. We have

$$\mathcal{A}(p) \cong \mathrm{Cl}_n(\mathbb{C}) \cong \mathrm{M}_k(\mathbb{C}) \oplus \mathrm{M}_k(\mathbb{C}),$$

and this algebra has now precisely two irreducible $*$-representations up to unitary equivalence, both of size k. They are for example given by the Brauer-Weyl matrices and their negatives. $\square$

We finish our work with two explicit examples for the above results.

Example 3.4.19. Consider $q_n = (x_1+\sqrt{2})^2 - x_2^2 - \cdots - x_n^2 - 1$. Writing $q_n = \underline{x}^t A\underline{x} + b^t\underline{x} + 1$ we see

$$\frac{1}{4}bb^t - A = I.$$

The above described homomorphism $\mathcal{A}(q_n) \to \mathrm{Cl}_n(\mathbb{C})$ is given by the rule

$$z_1 + J(q_n) \mapsto \sigma_1 + \sqrt{2}$$

$$z_i + J(q_n) \mapsto \sigma_i \quad \text{for } i = 2, \ldots, n.$$

We can substitute the Brauer-Weyl matrices (or their negatives) for the σ_j and obtain one or two different representations, depending on whether n is even or odd. Every other representation of some power is equivalent to a block sum of these minimal representations (and possibly trivial blocks, by Theorem 3.2.5). An explicit example of a minimal representation of q_5^2 is given by the following linear matrix polynomial:

$$\begin{pmatrix} 1+\sqrt{2}x_1+x_5 & x_1+ix_3 & x_2+ix_4 & 0 \\ x_1-ix_3 & 1+\sqrt{2}x_1-x_5 & 0 & -x_2-ix_4 \\ x_2-ix_4 & 0 & 1+\sqrt{2}x_1-x_5 & x_1+ix_3 \\ 0 & -x_2+ix_4 & x_1-ix_3 & 1+\sqrt{2}x_1+x_5 \end{pmatrix}.$$

Example 3.4.20. Consider $\hat{p}_n = (x_0+1)^2 - x_1^2 - \cdots - x_n^2$. Writing $\hat{p}_n = \underline{x}^t A\underline{x} + b^t\underline{x} + 1$ we see

$$\frac{1}{4}bb^t - A = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & I_n \end{array}\right),$$

and the homomorphism $\mathcal{A}(\hat{p}_n) \to \mathrm{Cl}_{n+1}(\mathbb{C})$ is given by the rule

$$z_0 + J(\hat{p}_n) \mapsto 1, \;\; z_i + J(\hat{p}_n) \mapsto \sigma_i \text{ for } i = 1, \ldots, n.$$

As above this leads to explicit representations, for example

$$\det\begin{pmatrix} 1+x_0 & x_1+ix_3 & x_2+ix_4 & 0 \\ x_1-ix_3 & 1+x_0 & 0 & -x_2-ix_4 \\ x_2-ix_4 & 0 & 1+x_0 & x_1+ix_3 \\ 0 & -x_2+ix_4 & x_1-ix_3 & 1+x_0 \end{pmatrix} = \hat{p}_4^2.$$

3.4.3 Rational Representations

We finish the first part of this work by a result on rational determinantal representations. If one allows for denominators, then in fact *every* real zero polynomial admits a symmetric linear determinantal representation. This is in particular interesting, since the representations provide an algebraic certificate for the geometric property of being real zero. On the other hand, the connection to spectrahedra is lost, when introducing denominators.

Let p be a square-free real zero polynomial. Then clearly p_a has only simple roots, for generic $a \in \mathbb{R}^n$. This implies that $\mathcal{H}(p)(a)$ is positive definite for generic a, by Proposition 2.2.4. $\mathcal{H}(p)$ is thus invertible over $\mathbb{R}(\underline{x})$. Let again $\mathcal{L}_t$ denote the companion matrix of p, as in Section 3.4.1.

Theorem 3.4.21. *Let $p \in \mathbb{R}[\underline{x}]$ be a square-free real zero polynomial. Let $q \in \mathbb{R}[\underline{x}] \setminus \{0\}$ be homogeneous and $Q \in \mathrm{M}_{k\times d}(\mathbb{R}[\underline{x}])$ as in Lemma 3.4.1, i.e. with*

$$q^2 \cdot \mathcal{H}(p) = Q^t Q.$$

Set

$$\mathcal{M} := q^{-2} Q \mathcal{L}_t \mathcal{H}(p)^{-1} Q^t.$$

Then $\mathcal{M}$ is symmetric, rational, homogeneous of degree 1, and

$$\det(I - \mathcal{M}) = p.$$

Proof. Abbreviate $\mathcal{H}(p)$ by H and $\mathcal{L}_t$ by L. By Sylvesters determinant criterion we have

$$\begin{aligned}\det(I_k - \mathcal{M}) &= \det(I_k - q^{-2} Q L H^{-1} Q^t) = \det(I_d - q^{-2} L H^{-1} Q^t Q)\\ &= \det(I_d - L) = p.\end{aligned}$$

We find

$$\mathcal{M}^t = q^{-2} Q H^{-t} L^t Q^t = q^{-2} Q L H^{-t} Q^t = \mathcal{M},$$

where we have used $L^t H = HL$, which is Lemma 3.4.2. So $\mathcal{M}$ is symmetric. Now let r denote the degree of q. By the degree structure of $q^2 \cdot H$ we find

$$Q(\lambda a) = Q(a) \cdot \mathrm{diag}(\lambda^r, \lambda^{r+1}, \ldots, \lambda^{r+d-1})$$

$$H(\lambda a) = \mathrm{diag}(\lambda^0, \ldots, \lambda^{d-1}) H(a) \mathrm{diag}(\lambda^0, \ldots, \lambda^{d-1})$$

$$L(\lambda a) = \mathrm{diag}(\lambda^d, \dots, \lambda^1)L(a)\mathrm{diag}(\lambda^{-d+1}, \lambda^{-d+2} \dots, \lambda^0)$$

for all $a \in \mathbb{R}^n$ and $\lambda \neq 0$. So for all a and λ for which the following expressions are defined, we have

$$\begin{aligned}
&\mathcal{M}(\lambda a)\\
&= \lambda^{-2r} q(a)^{-2} Q(a)\mathrm{diag}(\lambda^r, \dots, \lambda^{r+d-1})\mathrm{diag}(\lambda^d, \dots, \lambda)L(a)\mathrm{diag}(\lambda^{-d+1}, \dots, \lambda^0)\\
&\quad \cdot \mathrm{diag}(\lambda^0, \dots, \lambda^{-d+1})H^{-1}(a)\mathrm{diag}(\lambda^0, \dots, \lambda^{-d+1})\mathrm{diag}(\lambda^r, \dots, \lambda^{r+d-1})Q^t(a)\\
&= \lambda^{-2r} q(a)^{-2} Q(a)\lambda^{r+d} L(a)\lambda^{-d+1} H^{-1}(a)\lambda^r Q^t(a)\\
&= \lambda \cdot \mathcal{M}(a).
\end{aligned}$$

This shows that $\mathcal{M}$ is homogeneous of degree 1, and finishes the proof. □

Remark 3.4.22. In Theorem 3.4.21 we obtain a representation

$$p = \det(I - \mathcal{M})$$

for some symmetric $\mathcal{M}$ with $\mathcal{M}(\lambda a) = \lambda\mathcal{M}(a)$ for all a, λ. From such a representation it immediately follows that p is a real zero polynomial. So for each real zero polynomial there exists an algebraic certificate for the geometric real zero property.

Example 3.4.23. Consider $p = 1 - x_1^2 - \cdots - x_n^2$. We find

$$\mathcal{H}(p) = Q^t Q$$

with

$$Q = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2}x_1 \\ 0 & \vdots \\ 0 & \sqrt{2}x_n \end{pmatrix}.$$

This produces

$$\mathcal{M} = \begin{pmatrix} 0 & x_1 & \cdots & x_n \\ x_1 & & & \\ \vdots & & 0 & \\ x_n & & & \end{pmatrix}.$$

Example 3.4.24. Consider $p = (x_1+1)^2 - x_2^2 - x_3^2 - x_4^2$. We have

$$\mathcal{H}(p) = \begin{pmatrix} 2 & -2x_1 \\ -2x_1 & 2(x_1^2+x_2^2+x_3^2+x_4^2) \end{pmatrix} = Q^tQ$$

with

$$Q = \begin{pmatrix} \sqrt{2} & -\sqrt{2}x_1 \\ 0 & \sqrt{2}x_2 \\ 0 & \sqrt{2}x_3 \\ 0 & \sqrt{2}x_4 \end{pmatrix}.$$

We get

$$\mathcal{M} = \begin{pmatrix} -x_1 & x_2 & x_3 & x_4 \\ x_2 & -\frac{x_1x_2^2}{x_2^2+x_3^2+x_4^2} & -\frac{x_1x_2x_3}{x_2^2+x_3^2+x_4^2} & -\frac{x_1x_2x_4}{x_2^2+x_3^2+x_4^2} \\ x_3 & -\frac{x_1x_2x_3}{x_2^2+x_3^2+x_4^2} & -\frac{x_1x_3^2}{x_2^2+x_3^2+x_4^2} & -\frac{x_1x_3x_4}{x_2^2+x_3^2+x_4^2} \\ x_4 & -\frac{x_1x_2x_4}{x_2^2+x_3^2+x_4^2} & -\frac{x_1x_3x_4}{x_2^2+x_3^2+x_4^2} & -\frac{x_1x_4^2}{x_2^2+x_3^2+x_4^2} \end{pmatrix}.$$

Part II

Spectrahedral Shadows

Chapter 4

Definitions and First Results

4.1 Definitions

In the first part of this work we considered spectrahedra. Recall the definition: if $\mathcal{M} = M_0 + x_1 M_1 + \cdots + x_n M_n$ is a (hermitian or symmetric) linear matrix polynomial, then

$$\mathcal{S}(\mathcal{M}) = \{a \in \mathbb{R}^n \mid \mathcal{M}(a) \succeq 0\}$$

is a spectrahedron. Until now we assumed linear matrix polynomials to be monic, i.e. $M_0 = I$ to hold. This just meant restricting to spectrahedra containing the origin in the interior, as we have explained in Section 1.3. We will now drop this assumption, i.e. allow for arbitrary hermitian/symmetric matrices M_0.

We have seen that being a spectrahedron is quite a strong property for a set. Beyond being convex, closed and semi-algebraic, there are more necessary conditions. The most important one is rigid convexity, that was discussed in detail. It in fact already implies all the other properties stated in Section 1.3, like for example having exposed faces and being basic closed semi-algebraic.

We have also seen that each polyhedron is a spectrahedron. Now recall the well-known fact that any projection of a polyhedron, in fact any linear image of a polyhedron, is again a polyhedron. Interestingly, the same is not true for spectrahedra, as it turns out. We look at the following two easy examples.

Example 4.1.1. Consider

$$S = \{(a_1, a_2, a_3, a_4) \in \mathbb{R}^4 \mid a_1^2 \le a_3, a_2^2 \le a_4, a_3^2 + a_4^2 \le 1\}.$$

All three conditions defining S are easily seen to be spectrahedral, and so S is a spectrahedron. Note that S is even compact. Now consider the projection

$$\mathrm{p}\colon \mathbb{R}^4 \to \mathbb{R}^2; \quad (a_1, a_2, a_3, a_4) \mapsto (a_1, a_2).$$

We find

$$\mathrm{p}(S) = \{(a_1, a_2) \mid a_1^4 + a_2^2 \le 1\},$$

which is not a spectrahedron, as seen in Example 2.1.6.

Example 4.1.2. Consider

$$S = \{(a_1, a_2) \mid 0 \le a_1, 0 \le a_2, 1 \le a_1 a_2\}.$$

S is definable via the linear matrix polynomial

$$\mathcal{M} = \begin{pmatrix} a_1 & 1 \\ 1 & a_2 \end{pmatrix},$$

and is thus a spectrahedron. Note that S is not compact. If we now project S onto the first coordinate, we obtain the set

$$T = (0, \infty) \subseteq \mathbb{R}.$$

This set is not even closed, and thus not a spectrahedron.

From the viewpoint of optimization however, the linear image of a spectrahedron is still well manageable. Instead of optimizing a linear function over the image, one optimizes the pulled-back function (which is still linear) over the spectrahedron. One might however get a larger number of variables in the optimization problem.

So we make the following definition:

Definition 4.1.3. A set $S \subseteq \mathbb{R}^n$ is called a *spectrahedral shadow*, if there is an affine linear map $L\colon \mathbb{R}^N \to \mathbb{R}^n$ and a spectrahedron $T \subseteq \mathbb{R}^N$ with $S = L(T)$.

Remark 4.1.4. The name spectrahedral shadow is quite new, and seems to appear first in Rosalski and Sturmfels [51]. Earlier, spectrahedral shadows have been called *projections of spectrahedra* or *semidefinitely representable sets*. Quite often they are in fact defined as standard projections of spectrahedra only, i.e. images under a standard projection map

$$\mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n; (a, b) \mapsto a.$$

Spectrahedral shadows then have the following form:

$$\{a \in \mathbb{R}^n \mid \exists b \in \mathbb{R}^r \quad \mathcal{M}(a, b) \succeq 0\},$$

where $\mathcal{M}$ is a linear matrix polynomial. The following easy argument from Gouveia and Netzer [14] shows that passing from standard projections to arbitrary affine linear maps does not enlarge the class of sets.

Lemma 4.1.5. *If S has non-empty interior and is the image of a spectrahedron T under an affine linear map, then S is also a canonical projection of a spectrahedron T'. Furthermore, if T is definable by a strictly feasible linear matrix polynomial, then so is T'.*

Proof. Let $T = \{c \in \mathbb{R}^N \mid \mathcal{M}(c) \succeq 0\}$ be a spectrahedron, $L : \mathbb{R}^N \to \mathbb{R}^n$ an affine linear map and $S = L(T)$. We can furthermore restrict L to be linear, since translations of projections of spectrahedra are still projections of spectrahedra, even in the restricted sense. We also know that L is onto, since S has nonempty interior. By reordering the variables we can assume that $L = \begin{pmatrix} L_1 & L_2 \end{pmatrix}$, where L_1 is a $n \times n$ non-singular matrix. We can now consider the spectrahedron

$$T' = \left\{c \in \mathbb{R}^N \mid \mathcal{M}(L_1^{-1}(c_1, \ldots, c_n) - L_1^{-1}L_2(c_{n+1}, ..., c_m), c_{n+1}, ..., c_N) \succeq 0\right\}.$$

The projection of T' onto the first n variables equals S. Finally, if $\mathcal{M}$ is strictly feasible, then the defining linear matrix polynomial of T' is easily seen to be strictly feasible as well. $\square$

Note that there are two obvious properties of spectrahedral shadows: they are convex, and they are semi-algebraic. The first property is clear, the second follows from the so-called Projection Theorem for semi-algebraic sets, which is a consequence of quantifier elimination in the theory of real closed fields (explained for example in Prestel and Delzell [46]). So far, *no more* necessary conditions are known. In fact Helton and Nie [19, 20] conjecture that each convex semi-algebraic set is a spectrahedral shadow.

There are again certain elementary constructions that retain the property of being a spectrahedral shadow. For example, the intersection, the cartesian product and the Minkowski sum of two spectrahedral shadows is again a spectrahedral shadow. Also the inverse image of a spectrahedral shadow under an affine linear map is a spectrahedral shadow. All of these statements are easily checked. Some more complicated results are proven in the sequel.

We finish this section by some results that we will need in the following. They are also interesting for themselves. The first lemma is for example proven in Ramana and Goldman [47], Theorem 1. We sketch the easy proof.

Lemma 4.1.6. *If $f\colon \mathbb{R}^s \to \mathbb{R}^n$ is a quadratic polynomial map, then the convex hull of its image*

$$\mathrm{conv}\,(f(\mathbb{R}^s))$$

is a spectrahedral shadow.

Proof. First consider the map

$$f\colon \mathbb{R}^s \to \mathrm{Sym}_s(\mathbb{R}) \times \mathbb{R}^s, \quad a \mapsto (a^t a, a).$$

One checks that

$$\mathrm{conv}\,(f(\mathbb{R}^s)) = \left\{ (U,a) \mid \begin{pmatrix} U & a \\ a^t & 1 \end{pmatrix} \succeq 0 \right\}.$$

So this is even a spectrahedron. Now for arbitrary f, the convex hull of the image is just an affine linear image of $\mathrm{conv}\,(f(\mathbb{R}^s))$. □

For two symmetric matrices $A = (a_{ij}), B = (b_{ij}) \in \mathrm{Sym}_k(\mathbb{R})$ we define

$$A \bullet B := \mathrm{tr}(AB) = \sum_{i,j} a_{ij} b_{ij}$$

This defines an inner product on $\mathrm{Sym}_k(\mathbb{R})$. In [35], Section 4.1.1, Nemirovski proves the following result.

Proposition 4.1.7. *Let $\mathcal{M} = M_0 + x_1M_1 + \cdots + x_nM_n + y_1N_1 + \cdots + y_rN_r$ be a k-dimensional strictly feasible symmetric linear matrix polynomial. Let*

$$S := \{a \in \mathbb{R}^n \mid \exists b \in \mathbb{R}^r \ \mathcal{M}(a,b) \succeq 0\}$$

be the projection of the spectrahedron defined by $\mathcal{M}$, and let

$$S^\circ := \{\ell \in \mathbb{R}[\underline{x}]_1 \mid \ell \geq 0 \text{ on } S\}$$

denote the convex cone of affine linear polynomials nonnegative on S. Then

$$\begin{aligned} S^\circ = \{l_0 + l_1x_1 + \cdots + l_nx_n \mid \exists U \in \operatorname{Sym}_k(\mathbb{R}) : \quad & U \succeq 0,\ U \bullet M_0 \leq l_0, \\ & U \bullet M_i = l_i \text{ for } i = 1, \ldots, n, \\ & U \bullet N_j = 0 \text{ for } j = 1, \ldots, r\}. \end{aligned}$$

In particular, S° is again a spectrahedral shadow.

The result follows from the duality theory of conic programming, and is thus essentially a separation argument. The set S° is called the *polar* of S in Nemirovski's work. A key observation is that, for our purposes, the condition that $\mathcal{M}$ is strictly feasible is not necessary.

Lemma 4.1.8. *Let $S \subseteq \mathbb{R}^n$ be a spectrahedral shadow. Then*

$$S^\circ = \{\ell \in \mathbb{R}[\underline{x}]_1 \mid \ell \geq 0 \text{ on } S\}$$

is again a spectrahedral shadow.

Proof. Let S be the linear image of the spectrahedron T. We will first assume that S has nonempty interior in $\mathbb{R}^n$. By replacing the ambient space of T by the affine hull of T we can also assume that T has nonempty interior. Note that for a spectrahedron, having nonempty interior is equivalent to being definable by a strictly feasible linear matrix polynomial, as explained in Section 1.3. Lemma 4.1.5 now shows that we can apply Proposition 4.1.7, since S is the standard projection of a strictly feasible spectrahedron.

If S has empty interior, let V denote the affine hull of S. Then for $\ell \in \mathbb{R}[\underline{x}]_1$ the condition $\ell \geq 0$ on S is equivalent to $\ell|_V \geq 0$ on S. This shows that S° is the inverse image of a spectrahedral shadow under a linear map. Such sets are easily seen to be spectrahedral shadows as well. $\square$

In the next section we review the most important basic tools for checking whether a set is a spectrahedral shadow. They are mostly based on sums of squares representations of linear polynomials, and due to Lasserre, Parrilo et al.

4.2 Lasserre Relaxations

In this section we want to explain Lasserre's construction of spectrahedral shadows. Let again $\mathbb{R}[\underline{x}]$ denote the ring of real polynomials in n variables. By $\mathbb{R}[\underline{x}]_d$ we denote the finite dimensional vector space of polynomials of degree at most d. For a given finite tuple of polynomials $\underline{p} = (p_1, \ldots, p_m)$ we consider the basic closed semi-algebraic set defined by $\underline{p}$:

$$\mathcal{S}(\underline{p}) = \{a \in \mathbb{R}^n \mid p_1(a) \geq 0, \ldots, p_m(a) \geq 0\}\,.$$

Of course $\mathcal{S}(\underline{p})$ is not necessarily convex, but we are interested in checking whether the convex hull

$$\operatorname{conv}\left(\mathcal{S}(\underline{p})\right)$$

or the closed convex hull

$$\overline{\operatorname{conv}}\left(\mathcal{S}(\underline{p})\right)$$

is a spectrahedral shadow. The basic tool for this is due to Lasserre [27].

First consider the *quadratic module* $\mathrm{QM}(\underline{p})$ generated by $p_1, \ldots, p_m$. It consists of all polynomials one obtains from $p_1, \ldots, p_m$ by multiplying with sums of squares and adding. Formally, we define

$$\mathrm{QM}(\underline{p}) := \left\{\sigma_0 + \sigma_1 p_1 + \cdots + \sigma_m p_m \mid \sigma_i \in \sum \mathbb{R}[\underline{x}]^2\right\}.$$

Here $\sum \mathbb{R}[\underline{x}]^2$ denotes the set of sums of squares of polynomials. Note that $\mathrm{QM}(\underline{p})$ is a convex cone in the vector space $\mathbb{R}[\underline{x}]$. It is even closed under multiplication with squares of polynomials. Note also that all polynomials from $\mathrm{QM}(\underline{p})$ are clearly nonnegative as functions on $\mathcal{S}(\underline{p})$.

For our purpose we consider *truncated parts* of the quadratic module $\mathrm{QM}(\underline{p})$. Let $d \in \mathbb{N}$ be a nonnegative integer. Then define

$$\mathrm{QM}(\underline{p})_d := \left\{\sigma_0 + \sigma_1 p_1 + \cdots + \sigma_m p_m \in \mathrm{QM}(\underline{p}) \mid \deg(\sigma_0), \deg(\sigma_i p_i) \leq d\right\}.$$

So $\mathrm{QM}(\underline{p})_d$ is a convex cone in $\mathbb{R}[\underline{x}]_d$. It in fact consists of those elements of $\mathrm{QM}(\underline{p})$ which are *obviously* in $\mathbb{R}[\underline{x}]_d$, since each term $\sigma_i p_i$ in the representation is. It is the crucial point in the definition of $\mathrm{QM}(\underline{p})_d$ that the degree of each term is bounded, and not only the degree of the resulting polynomial. So $\mathrm{QM}(\underline{p})_d$ can and will be strictly smaller than $\mathrm{QM}(\underline{p}) \cap \mathbb{R}[\underline{x}]_d$ in general. We will see however that precisely the degree bounds make $\mathrm{QM}(\underline{p})_d$ useful for our purpose.

First we consider the dual cone of $\mathrm{QM}(\underline{p})_d$. It consists of all (normalized) linear forms on $\mathbb{R}[\underline{x}]_d$ that are nonnegative on $\mathrm{QM}(\underline{p})_d$:

$$\mathrm{QM}(\underline{p})_d^\vee := \left\{\varphi\colon \mathbb{R}[\underline{x}]_d \to \mathbb{R} \text{ linear} \mid \varphi \geq 0 \text{ on } \mathrm{QM}(\underline{p})_d, \varphi(1) = 1\right\}.$$

The dual space $\mathbb{R}[\underline{x}]_d^*$ of $\mathbb{R}[\underline{x}]_d$ is clearly a finite dimensional vector space. For example, when $\mathbb{R}[\underline{x}]_d$ is equipped with the basis consisting of all monomials

$$\underline{x}^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \text{ where } |\alpha| = \alpha_1 + \cdots + \alpha_n \leq d,$$

then each $\varphi \in \mathbb{R}[\underline{x}]_d^*$ can be identified with the tuple of its values on this basis:

$$\varphi = \left(\varphi(\underline{x}^\alpha)\right)_{|\alpha| \leq d}.$$

When making this identification, we see that $\mathrm{QM}(\underline{p})_d^\vee \subseteq \mathbb{R}[\underline{x}]_d^*$ is in fact a spectrahedron. This result has become quite common in the meantime. It is also explained in Lasserre [26], Marshall [32] or Schweighofer [57]. It uses the imposed degree bounds in a crucial way.

Lemma 4.2.1. *The set* $\mathrm{QM}(\underline{p})_d^\vee \subseteq \mathbb{R}[\underline{x}]_d^*$ *is a spectrahedron.*

Proof. We set $p_0 = 1$. Now an element $\varphi \in \mathbb{R}[\underline{x}]_d^*$ belongs to $\mathrm{QM}(\underline{p})_d^\vee$ if and only if $\varphi(1) = 1$ and

$$\varphi(h^2 p_i) \geq 0$$

for all $i = 0, 1, \ldots, m$ and all $h \in \mathbb{R}[\underline{x}]_{k_i}$, where k_i is the integer part of $\frac{1}{2}(d - \deg(p_i))$. The above condition can be transformed to

$$\sum_{|\alpha|,|\beta| \leq k_i} h_\alpha h_\beta \cdot \varphi\left(\underline{x}^{\alpha+\beta} \cdot p_i\right) \geq 0,$$

where the $h_\alpha \in \mathbb{R}$ are the coefficients of the polynomial h. So φ belongs to $\mathrm{QM}(\underline{p})_d^\vee$ if and only if $\varphi(1) = 1$ and the matrices

$$\left(\varphi(\underline{x}^{\alpha+\beta} \cdot p_i)\right)_{\alpha,\beta}$$

are positive semidefinite, for all i. Note that the rows and columns of the above matrix are understood to be indexed by all elements $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq k_i$. These matrices are clearly symmetric, and the entries are linear polynomials in the values $\varphi(\underline{x}^\alpha)$. This completes the proof. $\square$

Now Lasserre's idea from [27] is to project $\mathrm{QM}(\underline{p})_d^\vee$ to $\mathbb{R}^n$ by the rule

$$\varphi \mapsto (\varphi(x_1), \ldots, \varphi(x_n)) .$$

We denote the resulting set by $\mathcal{K}(\underline{p})_d$:

$$\mathcal{K}(\underline{p})_d := \left\{ (\varphi(x_1), \ldots, \varphi(x_n)) \mid \varphi \in \mathrm{QM}(\underline{p})_d^\vee \right\} .$$

This set is often called the d-th Lasserre relaxation of $\mathcal{S}(\underline{p})$. Lasserre shows that the spectrahedral shadows $\mathcal{K}(\underline{p})_d$ form a decreasing sequence of outer approximations of $\operatorname{conv}\left(\mathcal{S}(\underline{p})\right)$, and he gives sufficient conditions for this sequence to terminate at $\operatorname{conv}\left(\mathcal{S}(\underline{p})\right)$.

We however want to give a slightly different definition of the Lasserre relaxation, which seems more descriptive to us. We need some more work to explain our definition.

Lemma 4.2.2. *For any tuple $\underline{p} = (p_1, \ldots, p_m)$ of polynomials, the convex cone*

$$\mathrm{QM}(\underline{p})_d \subseteq \mathbb{R}[\underline{x}]_d$$

is a spectrahedral shadow.

Proof. We show that $\mathrm{QM}(\underline{p})_d$ is the image of some $\mathbb{R}^s$ under a quadratic polynomial map, and apply Lemma 4.1.6. Consider the mapping

$$\begin{aligned} \mathbb{R}[\underline{x}]_{k_0} \times \mathbb{R}[\underline{x}]_{k_1} \times \cdots \times \mathbb{R}[\underline{x}]_{k_m} &\to \mathbb{R}[\underline{x}]_d \\ (h_0, \ldots, h_m) &\mapsto h_0^2 + h_1^2 p_1 + \cdots + h_m^2 p_m. \end{aligned}$$

Here, the k_i are defined as in the proof of Lemma 4.2.1. By definition, $\mathrm{QM}(\underline{p})_d$ is the convex hull of the image of that map. □

Remark 4.2.3. One could also use Lemma 4.2.1 together with Lemma 4.1.8 to show that the double dual of $\mathrm{QM}(\underline{p})_d$ is a spectrahedral shadow. The double dual equals the closure, and quite often the cone $\mathrm{QM}(\underline{p})_d$ turns out to be close anyway (see for example the proof of Theorem 4.2.5).

We now give our modified definition of a Lasserre relaxation. It appears in Gouveia and Netzer [14]. For any $d \in \mathbb{N}$, it is the nonnegativity set of all linear polynomials having a sums of squares representation with degree bound d, i.e. belonging to $\mathrm{QM}(\underline{p})_d$.

Definition 4.2.4. Let $\underline{p} = (p_1, \dots, p_m)$ be a tuple of polynomials. Then

$$\mathcal{L}(\underline{p})_d := \left\{ a \in \mathbb{R}^n \mid \ell(a) \geq 0 \text{ for all } \ell \in \mathrm{QM}(\underline{p})_d \cap \mathbb{R}[\underline{x}]_1 \right\}$$

is called the *d-th Lasserre relaxation* of $\underline{p}$ (or slightly inexact, of $\mathcal{S}(\underline{p})$).

The following result subsumes the most important results concerning Lasserre relaxations. Parts (*iii*) and (*iv*) are like in Lasserre [27], part (*v*) is from Netzer, Plaumann and Schweighofer [38].

Theorem 4.2.5. *Let* $\underline{p} = (p_1, \dots, p_m)$ *be a tuple of polynomials and let* $S = \mathcal{S}(\underline{p})$. *Then the following is true:*

(1) Each set $\mathcal{L}(\underline{p})_d$ *is a closed spectrahedral shadow.*

(2) If S *has nonempty interior, one has* $\overline{\mathcal{K}(\underline{p})_d} = \mathcal{L}(\underline{p})_d$. *So our alternative relaxation coincides with the original one up to closures.*

(3) $\overline{\mathrm{conv}}\,(S) \subseteq \mathcal{L}(\underline{p})_{d+1} \subseteq \mathcal{L}(\underline{p})_d$ *holds for all* $d \in \mathbb{N}$.

(4) If $\mathrm{QM}(\underline{p})_d$ *contains all linear polynomials that are nonnegative on* S, *then* $\overline{\mathrm{conv}}\,(S) = \mathcal{L}(\underline{p})_d$. *In particular,* $\overline{\mathrm{conv}}\,(S)$ *is a spectrahedral shadow.*

(5) If S *has nonempty interior and* $\overline{\mathrm{conv}}(S) = \mathcal{L}(\underline{p})_d$, *then* $\mathrm{QM}(\underline{p})_d$ *contains all linear polynomials nonnegative on* S.

Proof. For (1) use Lemma 4.2.2 to see that

$$M_d := \mathrm{QM}(\underline{p})_d \cap \mathbb{R}[\underline{x}]_1$$

is a spectrahedral shadow. Thus by Lemma 4.1.8 also M_d° is a spectrahedral shadow. If we understand points $a \in \mathbb{R}^n$ as linear polynomials on $\mathbb{R}[\underline{x}]_1$, by the rule $\ell \mapsto \ell(a)$, we see that

$$M_d^\circ \cap \mathbb{R}^n = \mathcal{L}(\underline{p})_d$$

is also a spectrahedral shadow. Closedness of $\mathcal{L}(\underline{p})_d$ is clear.

For (2) let $\varphi \in \mathrm{QM}(\underline{p})_d^\vee$ and $\ell \in M_d$. Then

$$0 \leq \varphi(\ell) = \ell\left(\varphi(x_1), \dots, \varphi(x_n)\right),$$

which proves $\mathcal{K}(\underline{p})_d \subseteq \mathcal{L}(\underline{p})_d$, and thus the inclusion "$\subseteq$" from the claim. For "$\supseteq$" let $\ell \in \mathbb{R}[\underline{x}]_1$ be nonnegative on $\mathcal{K}(\underline{p})_d$. This means

$$0 \leq \ell\left(\varphi(x_1), \ldots, \varphi(x_n)\right) = \varphi(\ell)$$

for all $\varphi \in \mathrm{QM}(\underline{p})_d^{\vee}$. So ℓ belongs to the double dual cone of $\mathrm{QM}(\underline{p})_d$ (note that the condition $\varphi(1) = 1$ is non-restrictive; see the little trick in Marshall [31], proof of Theorem 3.1). The double dual cone is however well known to be the closure of $\mathrm{QM}(\underline{p})_d$. On the other hand, if S has nonempty interior, $\mathrm{QM}(\underline{p})_d$ is closed. This is proven for example in Marshall [32], Lemma 4.1.4, or Powers and Scheiderer [45], Proposition 2.6. So we get $\ell \in \mathrm{QM}(\underline{p})_d$, and this implies $\ell \geq 0$ on $\mathcal{L}(\underline{p})_d$. Since every linear polynomial that is nonnegative on $\mathcal{K}(\underline{p})_d$ is also nonnegative on $\mathcal{L}(\underline{p})_d$, this proves "$\supseteq$".

Statement (3) is clear, since every element from M_d is clearly nonnegative on S. Statement (4) is also clear. For (5) finally let $\ell \in \mathbb{R}[\underline{x}]_1$ be nonnegative on S. By assumption ℓ is also nonnegative on $\mathcal{L}(\underline{p})_d$. In fact we only need that ℓ is nonnegative on $\mathcal{K}(\underline{p})_d$. By arguing as in the proof of (2) we see $\ell \in \mathrm{QM}(\underline{p})_d$, using the closedness again. $\square$

Remark 4.2.6. The property that some $\mathrm{QM}(\underline{p})_d$ contains all linear polynomials nonnegative on S is called the *Putinar-Prestel bounded degree representation property for linear polynomials* in Lasserre [27]. In case the quadratic module is replaced by a preordering (which we don't consider here), it is called the *Schmüdgen bounded degree representation property.* Necessary geometric conditions for this property to hold are examined in Chapter 5 below.

A related but slightly different relaxation construction has been proposed in Gouveia, Parrilo and Thomas [15]. Their relaxations are called *theta body relaxations.* We roughly explain the idea. The starting point is not a basic closed semi-algebraic set $\mathcal{S}(\underline{p})$, but a real variety defined by some ideal $I \subseteq \mathbb{R}[\underline{x}]$. For fixed $d \in \mathbb{N}$ one considers the set of all polynomials that are a sums of squares of degree $2d$ modulo the ideal I:

$$\sum(d, I) = \left\{ f \in \mathbb{R}[\underline{x}] \mid f - \sigma \in I \text{ for some } \sigma \in \sum \mathbb{R}[\underline{x}]_d^2 \right\}.$$

The convex cone $\sum(d, I)$ is an analogue of $\mathrm{QM}(\underline{p})_{2d}$ from above. Now the d-th theta body of the real variety $\mathcal{V}_{\mathbb{R}}(I) \subseteq \mathbb{R}^n$ is

$$\mathrm{TH}(I)_d = \left\{ a \in \mathbb{R}^n \mid \ell(a) \geq 0 \text{ for all } \ell \in \sum(d, I) \cap \mathbb{R}[\underline{x}]_1 \right\}.$$

So the theta bodies are analogues of the Lasserre relaxations $\mathcal{L}(\underline{p})_d$. The theta bodies can be used to approximate the convex hull of $\mathcal{V}_{\mathbb{R}}(I)$ from above. A similar result as Theorem 4.2.5 (i), (iii), (iv) and (v) is true. The condition that S has nonempty interior has to be replaced by the condition that I is a real radical ideal. We will however not pursue this construction any further, and restrict to the Lasserre relaxations.

We finish this section with two explicit examples. The first example is from Lasserre [27].

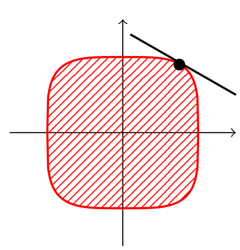
Example 4.2.7. Consider the set $S = \left\{(a_1, a_2 \in \mathbb{R}^2 \mid a_1^4 + a_2^4 \leq 1\right\}$. We have considered this set before in Example 2.1.6 and Example 4.1.1. We already know that S is a spectrahedral shadow, but not a spectrahedron. We will show how Lasserre's construction applies here, to give another justification that S is a spectrahedral shadow.

We consider the point $(a, \sqrt[4]{1-a^4})$, with $a \in [0,1]$ (it works similar for all other points on the boundary of S). Any linear polynomial nonnegative on S and zero at this point is a positive multiple of

$$\ell_a = 1 - a^3 x_1 - \left(\sqrt[4]{1-a^4}\right)^3 x_2.$$

See Figure 4.1 for a picture of S and the zero set of ℓ_a. One checks that $f := \ell_a - \lambda(1 - x_1^4 - x_2^4)$ is globally nonnegative, for some $\lambda > 0$. This can either be done elementary, or by the Karush-Kuhn-Tucker optimality conditions from convex optimization. Since f is globally nonnegative and of degree 4, it is a sums of squares. So $\ell_a \in \mathrm{QM}(1 - x_1^4 - x_2^4)_4$, and S is thus a spectrahedral shadow.

Figure 4.1:



The second example is due to João Gouveia, and published in Gouveia and Netzer [14].

Example 4.2.8. Let $\mathcal{S}(\underline{p}) \subseteq \mathbb{R}^2$ be defined by

$$p_1 = x_2, p_2 = 1 - x_2, p_3 = x_2 - x_1^3, p_4 = 1 + x_1.$$

We claim that $\mathcal{L}(\underline{p})_3$ is the convex hull of $\mathcal{S}(\underline{p}) \cup \{(1/3, 0)\}$. See Figure 4.2 for the sets $\mathcal{S}(\underline{p})$ and $\mathcal{L}(\underline{p})_3$. To prove the claim, let $C = \operatorname{conv}(\mathcal{S}(\underline{p}) \cup \{(1/3, 0)\})$. Then C is defined by the infinitely many affine linear inequalities

$$\{x_2 \geq 0, 1 - x_2 \geq 0, 1 + x_1 \geq 0, x_2 - 3a^2 x_1 + 2a^3 \geq 0 \mid a \in [1/2, 1]\},$$

since the polynomial $\ell_a := x_2 - 3a^2 x_1 + 2a^3$ defines the hyperplane tangent to the curve $x_2 = x_1^3$ at the point (a, a^3). To prove $\mathcal{L}(\underline{p})_3 \subseteq C$ it is thus enough to show that the polynomials ℓ_a belong to $\mathrm{QM}(\underline{p})_3$ for all $a \geq 1/2$. To see this, note that

$$\ell_a = (\sqrt{2a - 1}(x_1 - a))^2 + (x_2 - x_1^3) + (x_1 - a)^2 (x_1 + 1).$$

To prove the inclusion $C \subseteq \mathcal{L}(\underline{p})_3$, using the fact that $\mathcal{L}(\underline{p})_3$ is convex and contains $\mathcal{S}(\underline{p})$, it is enough to show that $(1/3, 0) \in \mathcal{L}(\underline{p})_3$. Since translations commute with taking Lasserre relaxations, we will instead consider the set of polynomials

$$p_1' = x_2,\ p_2' = 1 - x_2,\ p_3' = x_2 - x_1^3 - x_1^2 - \frac{1}{3}x_1 - \frac{1}{27},\ p_4' = x_1 + \frac{4}{3},$$

obtained from the p_i by replacing x_1 by $x_1 + 1/3$, and prove that $(0, 0) \in \mathcal{L}(\underline{p}')_3$. Suppose that is not the case. Then there must exist $\varepsilon, \mu > 0$ such that $\ell = x_2 - \mu x_1 - \varepsilon$ belongs to $\mathrm{QM}(\underline{p}')_3$. This means

$$\ell = \sigma_0 + \sigma_1 x_2 + \sigma_2 (1 - x_2) + c \left(x_2 - x_1^3 - x_1^2 - \frac{1}{3}x_1 - \frac{1}{27} \right) + \sigma_4 \left(x_1 + \frac{4}{3} \right),$$

where c is simply a nonnegative constant, since $\deg(p_3') = 3$. Note that $\sigma_0, \sigma_1, \sigma_2$ and σ_4 have at most degree 2.

Let $\sigma_4 = a_1 x_1^2 + a_2 x_1 + a_3 + a_4 x_2^2 + a_5 x_1 x_2 + a_6 x_2$. In order to cancel the x_1^3 term of the entire expression, we must have $a_1 = c$. The coefficient of x_1^2 will then be

$$a - c + \frac{4}{3}c + a_2,$$

where a is a nonnegative number, arising as the sum of the coefficients of x_1^2 in σ_0 and σ_2. This implies $a_2 \leq -c/3$, which by using the fact that σ_4 is a nonnegative polynomial, implies $a_3 \geq c/36$ (just set $x_2 = 0$ in σ_4 and use that the discriminant is nonpositive). Now the constant coefficient has to be

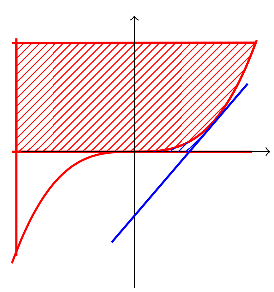
$$b - \frac{1}{27}c + \frac{4}{3}a_3,$$

where b is the nonnegative constant term of $\sigma_0 + \sigma_2$. Since this must be $-\varepsilon$, we have

$$-\frac{1}{27}c + \frac{4}{3}a_3 < 0,$$

which since $a_3 \geq c/36$ is impossible. Hence $\ell \notin \mathrm{QM}(\underline{p}')_3$, and $(1/3, 0)$ is in $\mathcal{L}(\underline{p})_3$ as intended. We see that $\mathcal{L}(\underline{p})_3$ does not coincide with $\mathcal{S}(\underline{p})$ in this example. It will follow from the results in Chapter 5 that indeed *none* of the Lasserre relaxations can coincide with $\mathcal{S}(\underline{p})$ here.

Figure 4.2:



4.3 Convex Hulls

In the last section we saw how Lasserre's method can show that convex hulls of basic closed semi-algebraic sets are spectrahedral shadows. This section contains two parts. First we show that the convex hull of the union of finitely many spectrahedral shadows is again a spectrahedral shadow. This generalizes and simplifies a result of Helton and Nie [19], where the additional assumption of boundedness was imposed upon the sets. The result appears in Netzer and Sinn [40]. In the second part we give a unified account of several results on convex hulls of images of polynomial maps. It summarizes and simplifies results that are spread across the literature. The overview also appears in Gouveia and Netzer [14].

We begin by proving that the class of spectrahedral shadows is closed under taking conic hulls.

Proposition 4.3.1. *If $S \subseteq \mathbb{R}^n$ is a spectrahedral shadow, then so is*

$$\operatorname{cone}(S) = \{\lambda \cdot a \mid a \in S, \lambda \geq 0\},$$

the conic hull of S.

Proof. We can assume that S has nonempty interior, and use Lemma 4.1.5 to write

$$S = \left\{ a \in \mathbb{R}^n \mid \exists b \in \mathbb{R}^r \colon M_0 + \sum_{i=1}^n a_i M_i + \sum_{j=1}^r b_j N_j \succeq 0 \right\},$$

with suitable hermitian or symmetric $k \times k$-matrices M_i, N_j. Then with

$$C := \{a \in \mathbb{R}^n \mid \exists \lambda, s \in \mathbb{R}, b \in \mathbb{R}^r \colon \lambda M_0 + \sum_{i=1}^n a_i M_i + \sum_{j=1}^r b_j N_j \succeq 0 \wedge \bigwedge_{i=1}^n \begin{pmatrix} \lambda & a_i \\ a_i & s \end{pmatrix} \succeq 0\}$$

we have $C = \operatorname{cone}(S)$ (note that C is a spectrahedral shadow).

To see "$\subseteq$" let some a fulfill all the conditions from C, first with some $\lambda > 0$. Then $\widetilde{a} := \frac{1}{\lambda} \cdot a$ belongs to S, using the first condition only. Since $a = \lambda \cdot \widetilde{a}$, $a \in \operatorname{cone}(S)$. If a fulfills the conditions with $\lambda = 0$, then $a = 0$, by the last n conditions in the definition of C. So clearly also $a \in \operatorname{cone}(S)$.

For "$\supseteq$" take $a \in \operatorname{cone}(S)$. If $a \neq 0$ then there is some $\lambda > 0$ and $\widetilde{a} \in S$ with $a = \lambda \cdot \widetilde{a}$. Now there is some $b \in \mathbb{R}^r$ with $M_0 + \sum_i \widetilde{a}_i M_i + \sum_j b_j N_j \succeq 0$. Multiplying this equation with λ shows that a fulfills the first condition in the definition of C. But since $\lambda > 0$, the other conditions can clearly also be satisfied with some big enough real s. So a belongs to C. Finally, $a = 0$ belongs to C too. $\square$

Remark 4.3.2. The additional n conditions in the definition of C avoid problems that could occur in the case $\lambda = 0$. This is the main difference to the approach of Helton and Nie [19].

Now the following corollary is straightforward:

Corollary 4.3.3. *If $S_1, \dots, S_t \subseteq \mathbb{R}^n$ are spectrahedral shadows, then the convex hull*

$$\operatorname{conv}(S_1 \cup \cdots \cup S_t)$$

is also a spectrahedral shadow.

Proof. Consider $\widetilde{S}_i := S_i \times \{1\} \subseteq \mathbb{R}^{n+1}$, and let K_i denote the conic hull of $\widetilde{S}_i$ in $\mathbb{R}^{n+1}$. All $\widetilde{S}_i$ and therefore all K_i are spectrahedral shadows. Thus the Minkowski sum $K := K_1 + \cdots + K_t$ is also a spectrahedral shadow. Now one easily checks

$$\operatorname{conv}(S_1 \cup \cdots \cup S_t) = \{a \in \mathbb{R}^n \mid (a, 1) \in K\},$$

which proves the result. $\square$

Example 4.3.4. Consider the set on the left in Figure 1.1 on page 17 in Section 1.3; it is the union of a disk and a square. We saw that is is not a spectrahedron, since failing to be basic closed semi-algebraic. However, since the disk and the square are spectrahedra, the set is a spectrahedral shadow.

Example 4.3.5. Let $S_1 := \{(a_1, a_2) \in \mathbb{R}^2 \mid a_1 \geq 0, a_2 \geq 0, a_1 a_2 \geq 1\}$ and $S_2 = \{(0, 0)\}$. Both subsets of $\mathbb{R}^2$ are spectrahedra, so the convex hull of their union,

$$\operatorname{conv}(S_1 \cup S_2) = \{(a_1, a_2) \in \mathbb{R}^2 \mid a_1 > 0, a_2 > 0\} \cup \{(0, 0)\},$$

is a spectrahedral shadow.

In the second part of this section we want to give a unified account of several results on convex hulls of images under polynomial maps, including results by Lasserre, Parrilo, Ramana and Goldman, Henrion and Scheiderer. The results can all be deduced from the following principle:

Proposition 4.3.6. *Let $T \subseteq \mathbb{R}^s$ be a set and $V \subseteq \mathbb{R}[y_1, \dots, y_s]$ a finite dimensional linear subspace containing 1. Assume the subset $P \subseteq V$ of all elements of V that are nonnegative on T is a spectrahedral shadow. Then for any map $f = (f_1, \dots, f_n) \colon \mathbb{R}^s \to \mathbb{R}^n$ with $f_i \in V$ for all i,*

$$\overline{\operatorname{conv}}(f(T)) \subseteq \mathbb{R}^n$$

is a spectrahedral shadow.

Proof. For any affine linear polynomial $\ell \in \mathbb{R}[x_1, \ldots, x_n]_1$ the polynomial $\ell(f_1, \ldots, f_n)$ belongs to V. Define

$$M := \{\ell \in \mathbb{R}[x_1, \ldots, x_n]_1 \mid \ell(f_1, \ldots, f_m) \in P\}.$$

M, as the inverse image of P under a linear map, is a spectrahedral shadow. It also contains only polynomials that are nonnegative on $f(T)$. Conversely, if ℓ is affine linear and nonnegative on $f(T)$, then $\ell(f_1, \ldots, f_n)$ is in P. Thus M is precisely the cone of affine linear polynomials nonnegative on $f(T)$. By the argument from the proof of Theorem 4.2.5 (1) (using Lemma 4.1.8) we find that

$$\overline{\mathrm{conv}(f(T))} = \{a \in \mathbb{R}^n \mid \ell(a) \geq 0 \text{ for all } \ell \in M\}$$

is a spectrahedral shadow. $\square$

Example 4.3.7. Not very surprisingly, Lasserre's result (Theorem 4.2.5 (4)) can be recovered from Proposition 4.3.6. Indeed if there is some d such that $\mathrm{QM}(\underline{p})_d$ contains all affine linear polynomials that are nonnegative on T, then apply Proposition 4.3.6 with $s = n$, $V = \mathbb{R}[\underline{x}]_1$ and $f = \mathrm{id}$. Then $P = V \cap \mathrm{QM}(\underline{p})_d$ is a spectrahedral shadow, by Lemma 4.2.2.

Example 4.3.8. We also get that the closure of $\mathrm{conv}(f(\mathbb{R}^s))$ is a spectrahedral shadow, for any quadratic map $f\colon \mathbb{R}^s \to \mathbb{R}^n$ (which is of course also not a new result, in view of Lemma 4.1.6 and Proposition 6.1.1 below). Use the well-known fact that every globally nonnegative quadratic polynomial is a sum of squares of affine linear polynomials, and apply Proposition 4.3.6 with $T = \mathbb{R}^s$ and $V = \mathbb{R}[y_1, \ldots, y_s]_2$. Again recall that $P = \sum \mathbb{R}[\underline{y}]_1^2 \subseteq V$ is a spectrahedral shadow.

In the following result, case (1) for a full rational curve is proven in Henrion [23], Theorem 1. In the version it is stated here it has also been the topic of a talk of Parrilo at a workshop in Banff in 2006, but there seems to be no suitable reference. Case (2) relies on results of Scheiderer, as also explained in [53].

Proposition 4.3.9. *Let $T \subseteq \mathbb{R}^s$ be either*

(1) a semi-algebraic subset of a rational curve, or

(2) a smooth curve of genus 1 with at least one non-real point at infinity.

Then for any rational map

$$f = \left(\frac{f_1}{g}, \dots, \frac{f_n}{g}\right) : \mathbb{R}^s \to \mathbb{R}^n$$

such that g does not vanish anywhere on T, we find that

$$\overline{\mathrm{conv}}(f(T))$$

is a spectrahedral shadow.

Proof. First check that we can reduce to the case where the denominator is 1, i.e. where f is a polynomial map. Indeed for a general rational map f we can take without loss of generality a denominator g that is positive on T, and we can also prove the claim for the following map instead:

$$F\colon T \to \mathbb{R}^{n+1};\ x \mapsto \left(\frac{f_1(x)}{g(x)}, \dots, \frac{f_n(x)}{g(x)}, 1\right).$$

Then define

$$G\colon T \to \mathbb{R}^{n+1};\ x \mapsto g(x) \cdot F(x).$$

This map is polynomial and thus assume we already know that $\overline{\mathrm{conv}}(G(T))$ is a spectrahedral shadow. By Proposition 4.3.1, the conic hull of a spectrahedral shadow is again a spectrahedral shadow. So together with Proposition 6.1.1 that we will prove in Section 6 we get that

$$\overline{\mathrm{cone}}(G(T)),$$

the closed conic hull of of $G(T)$, is a spectrahedral shadow. But now one can check, by a simple argument of converging sequences, that

$$\overline{\mathrm{cone}}(G(T)) \cap (\mathbb{R}^n \times \{1\}) = \overline{\mathrm{conv}}(F(T)),$$

which finishes the reduction step.

We now just have to show that in both cases the set of polynomials nonnegative on T of degree less or equal d is a spectrahedral shadow, for all d. We can then apply Proposition 4.3.6.

For (1) it is clearly enough to consider the case of T being a semi-algebraic subset of a straight line, which is covered by Kuhlmann, Marshall and Schwartz [25], Theorem 4.1. They prove that for any such set T and degree d, there is a d' and a suitable truncated quadratic module $\mathrm{QM}(\underline{p})_{d'}$ that contains all polynomials of degree at most d that are nonnegative on T.

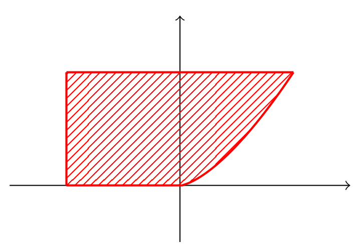
For (2), the results on the existence of sums of squares representations of nonnegative polynomials is Scheiderer [54], Theorem 4.10 (a). The degree bounds for these representations, as explained in Scheiderer [53], imply the intended result. So there is again a quadratic module containing all nonnegative polynomials of bounded degree in some truncated part. □

Example 4.3.10. The basic closed semi-algebraic set

$$S = \{(a_1, a_2) \in \mathbb{R}^2 \mid 0 \leq a_2 \leq 1, -1 \leq a_1, a_2^2 - a_1^3 \geq 0\}$$

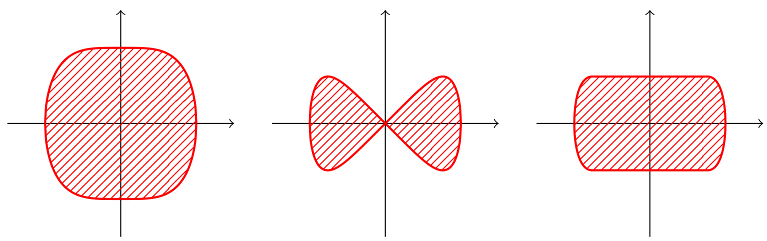
in Figure 4.3 is bounded by segments of rational curves. The convex hull of each such segment is a spectrahedral shadow, by Proposition 4.3.9. The set S is the convex hull of all of these convex hulls combined, and thus also a spectrahedral shadow, by Corollary 4.3.3. The standard Lasserre method does not apply to S directly, since S has a non-exposed face (see Theorem 5.2.1 in Chapter 5).

Figure 4.3:



Example 4.3.11. Let $S \subseteq \mathbb{R}^2$ be defined by the inequality $a_2^2 \leq 1 - a_1^4$, as shown on the left in Figure 4.4. The boundary of S is a smooth genus one curve with a non-real point at infinity. Thus S is a spectrahedral shadow. Applying the polynomial map $(a_1, a_2) \mapsto (a_1, a_1 a_2)$ sends this curve to the curve shown in the middle of Figure 4.4. This curve is clearly not smooth anymore. Still Proposition 4.3.9 guarantees that its convex hull is a spectrahedral shadow (shown on the right in Figure 4.4).

Figure 4.4:



We state some more corollaries of Proposition 4.3.6. The following result is Henrion [23], Theorem 1:

Corollary 4.3.12. *Let either $f\colon \mathbb{R}^3 \to \mathbb{R}^n$ be homogeneous of degree 4 or $f\colon \mathbb{R}^2 \to \mathbb{R}^n$ of degree 4, but not necessarily homogeneous. Then the closure of the convex hull of the image of f is a spectrahedral shadow.*

Proof. We can apply Proposition 4.3.6 together with Lemma 4.2.2, using Hilbert's result that every globally nonnegative homogeneous degree 4 polynomial in three variables, and every globally nonnegative degree 4 polynomial in two variables is a sum of squares. $\square$

We get another result that has not been observed before:

Corollary 4.3.13. *Let $f\colon \mathbb{R}^4 \to \mathbb{R}^n$ be homogeneous quadratic. Let $T \subseteq \mathbb{R}^4$ be any polyhedral cone. Then $\overline{\operatorname{conv}}(f(T))$ is a spectrahedral shadow.*

Proof. Every polyhedral cone in $\mathbb{R}^4$ is a finite union of cones that can be transformed by a linear automorphism to the first orthant in some $\mathbb{R}^k$, with $k \le 4$. This follows from Caratheodory's Theorem for cones. If

$$T = T_1 \cup \cdots \cup T_t$$

then

$$\overline{\operatorname{conv}}(f(T)) = \overline{\operatorname{conv}}\left(\overline{f(T_1)} \cup \cdots \cup \overline{f(T_t)}\right).$$

So by Corollary 4.3.3 and Proposition 6.1.1 below it is enough to prove the Corollary for the first orthant T in $\mathbb{R}^4$.

Every quadratic form in the 4 variables $y_1, \ldots, y_4$ that is nonnegative on the first orthant belongs to the quadratic module $\mathrm{QM}(\underline{p})$ generated by the pairwise products of the variables y_iy_j. This is just a slight reformulation of the main result from Diananda [11]. But then a degree bound condition on the sums of squares is fulfilled for any such representation, since no degree cancellation can occur when adding polynomials that are nonnegative on the first orthant. So in fact each such nonnegative quadratic form is a positive combination of the y_iy_j, plus a sums of squares of linear forms. Now apply Proposition 4.3.6 with V the space spanned by the quadratic forms and 1. Use that $P = \mathrm{QM}(\underline{p})_2 \cap V$ is a spectrahedral shadow. $\square$

4.4 The Results of Helton and Nie

Helton and Nie have proven several results on spectrahedral shadows in [19, 20]. Roughly, they apply the Lasserre relaxation method locally, and use the result on convex hulls (Lemma 4.3.3 in our work) to obtain a global result. Their papers contain several different results, some of them involving quite technical assumptions. We want to mention the result that seems most important and least technical to us. It is Theorem 2 in [20]. We need the following definitions:

Definition 4.4.1. (1) Let $p \in \mathbb{R}[\underline{x}]$. Then p is called *sos-concave*, if its negative Hessian matrix $-\nabla^2(p)$ is a sum of squares, i.e. if there is a matrix $W \in \mathrm{M}_{l\times n}(\mathbb{R}[\underline{x}])$ with

$$-\nabla^2(p) = W^tW.$$

(2) Let $S \subseteq \mathbb{R}^n$ be a set. Then $p \in \mathbb{R}[\underline{x}]$ is called *strictly quasi concave on S*, if the Hessian of p is negative definite on the orthogonal complement of the gradient of p, at each point of S, i.e. if

$$v^t\nabla^2(p)(a)v < 0 \text{ for all } 0 \neq v \in \nabla(p)(a)^\perp.$$

Now the result of Helton and Nie is the following:

Theorem 4.4.2 (Helton & Nie [20])**.** *Let $\underline{p} = (p_1, \ldots, p_m)$ be such that $\mathcal{S}(\underline{p})$ is compact and convex with nonempty interior. Assume that each p_i is either sos-concave, or strictly quasi concave on $\mathcal{S}(\underline{p})$. Then $\mathcal{S}(\underline{p})$ is a spectrahedral shadow.*

Example 4.4.3. We can apply the result to the set

$$S = \{(a_1, a_2) \in \mathbb{R}^2 \mid a_1^4 + a_2^4 \leq 1\}.$$

The defining polynomial $p = 1 - x_1^4 - x_2^4$ turns out to be sos-concave. Indeed we have

$$-\nabla^2(p) = \begin{pmatrix} 12x_1^2 & 0 \\ 0 & 12x_2^2 \end{pmatrix} = \begin{pmatrix} \sqrt{12}x_1 & 0 \\ 0 & \sqrt{12}x_2 \end{pmatrix}^t \begin{pmatrix} \sqrt{12}x_1 & 0 \\ 0 & \sqrt{12}x_2 \end{pmatrix}.$$

Example 4.4.4. Let $p_1 = 2 - x_1, p_2 = x_1 - 1, p_3 = 1 - x_2, p_4 = x_1^2 x_2 - 1$. The set $\mathcal{S}(\underline{p})$ is shown in Figure 4.5. The polynomials p_1, p_2, p_3 have a trivial Hessian matrix, and are thus sos-concave. The Hessian of p_4 at some point $a = (a_1, a_2) \in \mathcal{S}(\underline{p})$ is

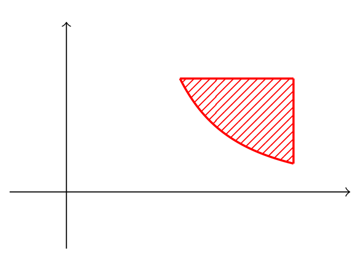
$$\nabla^2(p_4)(a) = \begin{pmatrix} 2a_2 & 2a_1 \\ 2a_1 & 0 \end{pmatrix}.$$

The orthogonal complement of $\nabla(p_4)(a)$ is spanned by $\begin{pmatrix} a_1 \\ -2a_2 \end{pmatrix}$. We thus compute

$$-(a_1, -2a_2)\nabla^2(p_4)(a) \begin{pmatrix} a_1 \\ -2a_2 \end{pmatrix} = 6a_1^2 a_2.$$

This expression is clearly positive on $\mathcal{S}(\underline{p})$. So p_4 is strictly quasi concave on $\mathcal{S}(\underline{p})$, and $\mathcal{S}(\underline{p})$ is a spectrahedral shadow.

Figure 4.5:



Chapter 5

Limitations of the Relaxation Methods

5.1 Preliminaries

In this chapter we examine the Lasserre relaxation method in more detail. We derive necessary conditions for it to work. Our main result will show that whenever a convex set $\mathcal{S}(\underline{p})$ with nonempty interior has a non-exposed face, no Lasserre relaxation can be exact, i.e.

$$\mathcal{S}(\underline{p}) \subsetneq \mathcal{L}(\underline{p})_d$$

holds for all $d \in \mathbb{N}$. This resembles a property of spectrahedra: they have only exposed faces. Note however that a convex set with a non-exposed face can of course be a spectrahedral shadow. We have already seen examples, e.g. Example 4.3.4 and Example 4.3.10. Only the *global* Lasserre method cannot work in these cases. Note also that Theorem 4.2.5 provides an alternative formulation for our main result: if $\mathcal{S}(\underline{p})$ has a non-exposed face, then not all nonnegative linear polynomials can have a bounded degree sums of squares representation in $\mathrm{QM}(\underline{p})_d$. This formulation is purely real-algebraic, and does not involve the theory of spectrahedral shadows. The result is published in Netzer, Plaumann and Schweighofer [38].

Before we start, we repeat some important notions. A *face* of a convex set $S \subseteq \mathbb{R}^n$ is a nonempty convex subset $F \subseteq S$, such that for $x, y \in S$ and $\lambda \in [0,1]$, $\lambda x + (1-\lambda)y \in F$ implies $x, y \in F$. A special case is that of an extreme point (i.e. if $F = \{a\}$ is a singleton). Note that there can only be finite increasing chains of faces

$$\cdots \subsetneq F' \subsetneq F'' \subsetneq \cdots$$

of S. Indeed, the dimension of the affine hull of a face strictly grows if the face becomes larger. We will in the following denote by $\dim(S)$ the dimension of the affine hull of a convex set S.

A face F of S is *exposed*, if there is an affine linear function ℓ on $\mathbb{R}^n$, with $\ell \geq 0$ on S and

$$F = \{a \in S \mid \ell(a) = 0\}.$$

In case that $F \neq S$ this is the same as saying that there is a supporting hyperplane of S that touches S precisely in F.

We need some technical lemmas first. The following is a special case of Proposition II.5.16 in Alfsen [3], equipped with an alternative proof.

Lemma 5.1.1. *Let S be a closed convex subset of $\mathbb{R}^n$. A face F of S is exposed if and only if for every $a \in S \setminus F$ there exists a supporting hyperplane H_a of S with $F \subseteq H_a$ and $a \notin H_a$.*

Proof. Necessity is obvious. For sufficiency, write $F_a = S \cap H_a$. Each F_a is an exposed face, containing F. By assumption we find

$$F = \bigcap_{a \in S \setminus F} F_a.$$

Since an arbitrary nonempty intersection of faces is again a face, and there can be no infinite chains of faces, this intersection is in fact finite:

$$F = \bigcap_{i=1}^{t} F_{a_i}.$$

So if ℓ_i is an affine linear polynomial nonnegative on S with

$$F_{a_i} = \{a \in S \mid \ell_i(a) = 0\},$$

then the sum $\ell = \sum_{i=1}^t \ell_i$ exposes F as a face of S. □

Lemma 5.1.2. *Let S be a closed convex subset of $\mathbb{R}^n$ with non-empty interior. A face F of S is exposed if and only if $F \cap U$ is an exposed face of $S \cap U$, for every affine linear subspace U of $\mathbb{R}^n$ containing F with*

$$\dim(U) = \dim(F) + 2$$

and $U \cap \operatorname{int}(S) \neq \emptyset$.

Proof. Note first that the condition is empty if F is of dimension $\geqslant n-1$. Indeed, F is always exposed in that case, as is easily checked. Thus we may assume that $n \geqslant 2$ and $\dim(F) \leqslant n-2$.

If H exposes F and $U \cap \operatorname{int}(S)$ is non-empty, then $H \cap U$ exposes F in $S \cap U$. Conversely, assume that $F \cap U$ is an exposed face of $S \cap U$ for every U satisfying the hypothesis. We want to apply the preceding lemma. For $a \in S \setminus F$ we must produce a supporting hyperplane H of S containing F, with $a \notin H$. Choose U to be an affine linear subspace of $\mathbb{R}^n$ of dimension $\dim(F)+2$ containing F, such that $a \in U$ and $U \cap \operatorname{int}(S) \neq \emptyset$. By hypothesis, there exists a supporting hyperplane G of $S \cap U$ in U that exposes F as a face of $S \cap U$. In particular, $a \notin G$. Since $G \cap S = F$, it follows that $G \cap \operatorname{int}(S) = \emptyset$, hence by separation of disjoint convex sets (see e.g. Barvinok [4], Thm. III.1.2), there exists a hyperplane H that satisfies $G \subseteq H$ and $H \cap \operatorname{int}(S) = \emptyset$. Since $U \cap \operatorname{int}(S) \neq \emptyset$, it follows that $G \subseteq H \cap U \subsetneq U$, hence $G = H \cap U$. Thus H is a supporting hyperplane of S containing F with $a \notin H$. $\square$

We finally need the following technical lemma.

Lemma 5.1.3. *Let S be a convex subset and U an affine linear subspace of $\mathbb{R}^n$ intersecting the interior of S. Suppose that $\ell\colon \mathbb{R}^n \to \mathbb{R}$ is an affine linear function such that $\ell \geqslant 0$ on $S \cap U$. Then there exists an affine linear function $\ell'\colon \mathbb{R}^n \to \mathbb{R}$ such that $\ell' \geqslant 0$ on S and $\ell'|_U = \ell|_U$.*

Proof. Let $N := \{x \in U \mid \ell(x) < 0\}$ and let S' be the convex hull of

$$\{x \in U \mid \ell(x) \geqslant 0\} \cup S.$$

Then N and S' are convex sets that we now prove to be disjoint. Assume for a contradiction that there are $\lambda \in [0,1]$, $x \in U$ and $y \in S$ such that $\ell(x) \geqslant 0$ and $\lambda x + (1-\lambda)y \in N$. Since neither x nor y lies in N, we have $\lambda \notin \{0,1\}$. Since U is an affine linear subspace, $\lambda x + (1-\lambda)y \in U$ now implies $y \in U$ and therefore $\ell(y) \geqslant 0$, leading to the contradiction

$$0 > \ell(\lambda x + (1-\lambda)y) = \lambda \ell(x) + (1-\lambda)\ell(y) \geqslant 0.$$

Without loss of generality we can assume $N \neq \emptyset$ (otherwise $\ell|_U = 0$ and we can take $\ell' = 0$). Then by separation of non-empty disjoint convex sets (e.g. Thm. III.1.2 in Barvinok [4]), we get an affine linear function

$\ell' \colon \mathbb{R}^n \to \mathbb{R}$, not identically zero, such that $\ell' \geqslant 0$ on S' and $\ell' \leqslant 0$ on N. In particular, $\ell' \geqslant 0$ on S and ℓ' cannot vanish at an interior point of S. Since U intersects by hypothesis the interior of S, it is not possible that ℓ' vanishes identically on U. Moreover, all $x \in U$ with $\ell(x) = 0$ lie at the same time in S' and in the closure of N, implying that $\ell'(x) = 0$. This shows that the restrictions of ℓ and ℓ' on U are the same up to a positive factor, which we may assume to be 1 after rescaling. □

5.2 Main Result

We are now ready for the main result from Netzer, Plaumann and Schweighofer [38]. It gives a necessary condition for the bounded degree representation property to hold.

Theorem 5.2.1. *Let $S = \mathcal{S}(\underline{p}) \subseteq \mathbb{R}^n$ be convex with non-empty interior. Suppose that there exists $d \geqslant 1$ such that the d-th Lasserre relaxation is exact, i.e.*

$$\mathcal{L}(\underline{p})_d = S$$

holds. Then all faces of S are exposed.

In view of Theorem 4.2.5 (*iv*) and (*v*), we have the following equivalent formulation of the same theorem:

Theorem (Alternative formulation)**.** *Let $S = \mathcal{S}(\underline{p}) \subseteq \mathbb{R}^n$ be convex with non-empty interior. Suppose that there exists $d \geqslant 1$ such that every linear polynomial ℓ with $\ell \geqslant 0$ on S is contained in $\mathrm{QM}(\underline{p})_d$. Then all faces of S are exposed.*

Proof. We begin by showing that it is sufficient to prove that all faces of dimension $n-2$ are exposed. Let F be a face of S of dimension e. For $e \geqslant n-1$ there is nothing to show, so assume $e \leqslant n-2$. If F is not exposed, then by Lemma 5.1.2 there exists an affine linear subspace U of $\mathbb{R}^n$ containing F, with $\dim(U) = e+2$ and $U \cap \mathrm{int}(S) \neq \emptyset$, and such that F is a non-exposed face of $S \cap U$. Furthermore, by Lemma 5.1.3, for every linear polynomial ℓ that is nonnegative on $S \cap U$ there exists a linear polynomial ℓ' that is nonnegative on S and agrees with ℓ on U. Upon replacing $\mathbb{R}^n$ by U and S by $S \cap U$, we reduce to the case $e = n-2$.

Now assume for contradiction that $d \geqslant 1$ as in the statement exists and that F is a face of dimension $n-2$ that is not exposed.

Step 1. There is exactly one supporting hyperplane H of S that contains F. For if ℓ_1, ℓ_2 are non-zero linear polynomials with $\ell_i|_F = 0$ and $\ell_i|_S \geqslant 0$, put $W := \{\ell_1 = 0\} \cap \{\ell_2 = 0\}$. Then $\ell := \ell_1 + \ell_2$ defines a supporting hyperplane $\{\ell = 0\}$ of S with $\{\ell = 0\} \cap S = W \cap S$. If ℓ_1, ℓ_2 are linearly independent, then $\dim(W) = n - 2 = \dim(F)$, hence $F = \{\ell = 0\} \cap S$, contradicting the fact that F is not exposed.

We may assume after an affine change of coordinates that

$$H = \{x_1 = 0\}, \quad x_1 \geqslant 0 \text{ on } S,$$

and that 0 lies in the relative interior of F. Note that any supporting hyperplane of S containing 0 must contain F and therefore coincide with H.

Since F is not exposed, $F_0 = H \cap S$ is a face of dimension $n-1$ with F contained in its relative boundary. In particular, it follows that F is also contained in the closure of $\partial S \setminus H$.

Step 2. By the curve selection lemma (see e.g. Thm. 2.5.5. in Bochnak, Coste, and Roy [7]), we may choose a continuous semi-algebraic path

$$\gamma\colon [0,1] \to \partial S$$

such that $\gamma(0) = 0 \in F$, $\gamma\big((0,1]\big) \cap H = \emptyset$. We relabel $1 = p_0, p_1, \ldots, p_m$ into two groups $f_1, \ldots, f_r$, $g_1, \ldots, g_s$ as follows:

$$\begin{aligned} f_i|_{\gamma([0,1])} &= 0 \quad (i = 1, \ldots, r) \\ g_j|_{\gamma((0,1])} &> 0 \quad (j = 1, \ldots, s) \end{aligned}$$

Indeed, after restricting γ to $[0, \alpha]$ for suitable $\alpha \in (0,1]$ and re-parametrizing, we can assume that each p_i falls into one of the above categories.

We claim that there exists an expression

$$(*) \qquad x_1 = \sum_{i=1}^{r} \rho_i f_i + \sum_{j=1}^{s} \sigma_j g_j$$

with $\rho_i, \sigma_j \in \sum \mathbb{R}[\underline{x}]^2$ and such that $\sigma_j(0) = 0$ for all $j = 1, \ldots, s$.

To prove the existence of the expression $(*)$, consider the following statement:

(†) For each $\lambda \in (0,1]$ there exists a linear polynomial $\ell_\lambda \in \mathbb{R}[\underline{x}]_1$ such that $\ell_\lambda(\gamma(\lambda)) = 0$, $\ell_\lambda \geqslant 0$ on S, and $||\ell_\lambda|| = 1$. For this ℓ_λ, there exist $\rho_i^{(\lambda)}, \sigma_j^{(\lambda)} \in \sum \mathbb{R}[\underline{x}]_d^2$ such that

$$\ell_\lambda = \sum_{i=1}^{r} \rho_i^{(\lambda)} f_i + \sum_{j=1}^{s} \sigma_j^{(\lambda)} g_j$$

and such that

$$\sigma_j^{(\lambda)}(\gamma(\lambda)) = 0$$

for all $j = 1, \ldots, s$.

The statement (†) is true, with $d \geqslant 1$ not depending on λ: For $\lambda \in (0,1]$, let $\ell_\lambda \in \mathbb{R}[\underline{x}]_1$ be such that $\{\ell_\lambda = 0\}$ is a supporting hyperplane of S passing through $\gamma(\lambda)$, and such that $||\ell_\lambda|| = 1$ and $\ell_\lambda|_S \geqslant 0$. By hypothesis,

$$\ell_\lambda \in \mathrm{QM}(f_1, \ldots, f_r, g_1, \ldots, g_s)_d,$$

with d not depending on λ, which yields the desired representation. Note that $\sigma_j^{(\lambda)}(\gamma(\lambda)) = 0$ is automatic, since $g_j(\gamma(\lambda)) \neq 0$, but $\ell_\lambda(\gamma(\lambda)) = 0$.

The crucial point is now that (†) can be expressed as a first-order formula in the language of ordered rings. The bound d on the degree of the sums of squares automatically also bounds their lengths, by Carathéordory's Theorem. So the existence of the representation (†) is in fact the existence of finitely many coefficients of polynomials.

Thus (†) holds over any real closed extension field R of $\mathbb{R}$, by the model-completeness of the theory of real closed fields. Now let R be any proper (hence non-archimedean) extension field and let $\varepsilon \in R$, $\varepsilon > 0$, be an infinitesimal element with respect to $\mathbb{R}$. We apply (†) with $\lambda = \varepsilon$ and get

$$\ell_\varepsilon = \sum_{i=1}^{r} \rho_i^{(\varepsilon)} f_i + \sum_{j=1}^{s} \sigma_j^{(\varepsilon)} g_j \tag{‡}$$

with

$$\sigma_j^{(\varepsilon)}(\gamma(\varepsilon)) = 0$$

for all $j = 1, \ldots, s$. Let $\mathcal{O}$ be the convex hull of $\mathbb{R}$ in R, a valuation ring with maximal ideal $\mathfrak{m}$. Since $\mathrm{int}(S) \neq \emptyset$, the quadratic module

$$M = \mathrm{QM}(f_1, \ldots, f_r, g_1, \ldots, g_r) \subseteq \mathbb{R}[\underline{x}]$$

has trivial support (which means $M \cap -M = \{0\}$, i.e. M is a pointed cone). As $||\ell_\varepsilon|| = 1$, it follows that all coefficients of the polynomials in (‡) must lie in $\mathcal{O}$ (see e.g. the proof of Lemma 8.2.3 in Prestel and Delzell [46]). We can therefore apply the residue map

$$\mathcal{O} \to \mathcal{O}/\mathfrak{m} \cong \mathbb{R},\ a \mapsto \overline{a}$$

to the coefficients of (‡). From the uniqueness of the supporting hyperplane $H = \{x_1 = 0\}$ in 0 (Step 1), it follows that $\overline{\ell_\varepsilon} = c \cdot x_1$ for some $c \in \mathbb{R}_{>0}$. This yields the desired expression (∗).

Step 3. The existence of (∗) leads to a contradiction: Substituting $x_1 = 0$ in (∗) gives

$$0 = \sum_{i=1}^{r} \rho_i(0, \underline{x}')f_i(0, \underline{x}') + \sum_{j=1}^{s} \sigma_j(0, \underline{x}')g_j(0, \underline{x}')$$

in $\mathbb{R}[\underline{x}']$, with $\underline{x}' = (x_2, \dots, x_n)$. Since all $f_i(0, \underline{x}'), g_j(0, \underline{x}')$ are non-negative on F_0, which has non-empty interior in H, it follows that $\rho_i(0, \underline{x}') = 0$ whenever $f_i(0, \underline{x}') \neq 0$. In other words, if x_1 does not divide f_i, then x_1^2 divides ρ_i in $\mathbb{R}[\underline{x}]$.

Going back to (∗) and substituting $x_2 = \cdots = x_n = 0$ now gives

$$x_1 = \sum_{i=1}^{r} \rho_i(x_1, 0)f_i(x_1, 0) + \sum_{j=1}^{s} \sigma_j(x_1, 0)g_j(x_1, 0)$$

Since $\sigma_j(0) = 0$ for all $j = 1, \dots, s$, we now know that x_1^2 divides all terms on the right-hand side, except possibly $\rho_i(x_1, 0)f_i(x_1, 0)$ for such i where x_1 divides f_i. In the latter case, write $f_i = x_1\widetilde{f_i}$ and note that $\widetilde{f_i}$ vanishes on $\gamma((0, 1])$ since f_i does and x_1 does not. Thus $\widetilde{f_i}(0) = 0$ by continuity, which implies $x_1|\widetilde{f_i}(x_1, 0)$, so $x_1^2|f_i(x_1, 0)$ after all. It follows that x_1^2 divides x_1, a contradiction. □

Remark 5.2.2. (1) Whether the faces of $S = \mathcal{S}(\underline{p})$ are exposed is a purely geometric condition, independent of the choice of the polynomials $\underline{p}$. Thus if S has a non-exposed face, there do not exist polynomials $\underline{p}$ defining S that yield an exact Lasserre relaxation for S.

(2) The theorem does *not* imply that a basic closed semi-algebraic convex set with a non-exposed face cannot be a spectrahedral shadow, as we have already mentioned. We only shown that Lasserre's global relaxation approach can not work in that case.

5.3 Examples

We conclude the chapter with some examples.

Example 5.3.1. Consider again the set $\mathcal{S}(\underline{p}) \subseteq \mathbb{R}^2$ from Example 4.2.8, defined by

$$p_1 = x_2, p_2 = 1 - x_2, p_3 = x_2 - x_1^3, p_4 = 1 + x_1,$$

as shown in Figure 4.2 on page 91. We saw that the third Lasserre relaxation does approximate $\mathcal{S}(\underline{p})$, but is not exact. Since the origin is a non-exposed face of $\mathcal{S}(\underline{p})$, *no* Lasserre relaxation is exact. Another way to state this is that there do not exist polynomials $\underline{q}$ with $\mathcal{S}(\underline{p}) = \mathcal{S}(\underline{q})$, such that all linear polynomials that are nonnegative on $\mathcal{S}(\underline{q})$ belong to $\mathrm{QM}(\underline{q})_d$ for some *fixed* value of d.

On the other hand, the preordering generated by p_1, p_2, p_3, p_4 as above (i.e. the quadratic module generated by all products of the p_i) contains all polynomials that are nonnegative on $\mathcal{S}(\underline{p})$. This follows from results of Scheiderer. Indeed, by the local-global principle Corollary 2.10 from Scheiderer [55] it suffices to show that the preordering generated by the p_i is locally saturated. At the origin this follows from the results in Scheiderer [56] (in particular, Theorem 6.3 and Corollary 6.7). At all other points it follows already from [55], Lemma 3.1.

However, from the Lemma 4.3.3, we can deduce that $\mathcal{S}(\underline{p})$ is in fact a spectrahedral shadow: For $\mathcal{S}(\underline{p})$ is the (convex hull of the) union of the sets $S_1 = [-1, 0] \times [0, 1]$ and $S_2 = \mathcal{S}(x_2 - x_1^3, x_1, 1 - x_2)$. The set S_1 is obviously a spectrahedral shadow (even a spectrahedron), while S_2 possesses an exact Lasserre relaxation: More precisely, we claim that

$$\mathrm{QM}(x_2 - x_1^3, x_1, 1 - x_2)_3$$

contains all linear polynomials $\ell \in \mathbb{R}[x_1, x_2]_1$ such that $\ell|_{S_2} \geqslant 0$. It suffices to show this for the tangents $\ell_a = x_2 - 3a^2 x_1 + 2a^3$ to S_2 passing through the points (a, a^3), $a \in [0, 1]$ (the claim then follows from Farkas's lemma). Write $\ell_a = x_1^3 - 3a^2 x_1 + 2a^3 + (x_2 - x_1^3)$. The polynomial $x_1^3 - 3a^2 x_1 + 2a^3 \in \mathbb{R}[x_1]$ is non-negative on $[0, \infty)$ and therefore contained in $\mathrm{QM}(x_1)_3 \subseteq \mathbb{R}[x_1]$ (see Kuhlmann, Marshall, and Schwartz [25], Theorem. 4.1). This implies the claim.

In Netzer, Plaumann and Schweighofer [38] it was asked whether any Lasserre relaxation $\mathcal{L}(\underline{p})_d$ has only exposed faces. This is false, as shows for instance Example 4.2.8, due to João Gouveia. The relaxation $\mathcal{L}(\underline{p})_3$ has a non-exposed extreme point. It was also asked whether Theorem 5.2.1 remains true for non-convex sets $\mathcal{S}(\underline{p})$, i.e. is

$$\mathcal{L}(\underline{p})_d = \overline{\text{conv}}\left(\mathcal{S}(\underline{p})\right)$$

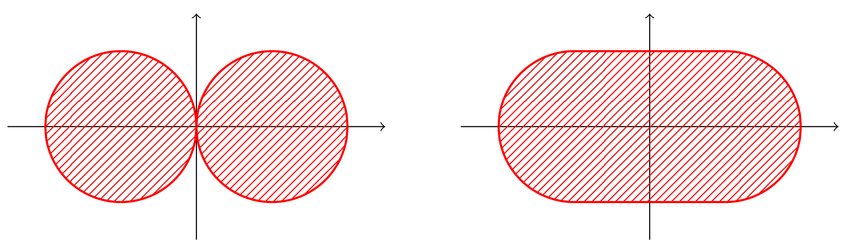
only possible if $\overline{\text{conv}}\left(\mathcal{S}(\underline{p})\right)$ has all exposed faces? João Gouveia showed that also this is false, by giving the following example (published in Gouveia and Netzer [14]).

Example 5.3.2. For $p := -x_1^4 - x_2^4 - 2x_1^2x_2^2 + 4x_1^2 \in \mathbb{R}[x_1, x_2]$ we find

$$\mathcal{L}(p)_4 = \text{conv}\left(\mathcal{S}(p)\right).$$

Indeed the set $S = \mathcal{S}(p)$ is the union of two disks of radius 1 with centers $(-1, 0)$ and $(1, 0)$, as shown in Figure 5.1 on the left.

Figure 5.1:



By symmetry, it is enough to show that any linear polynomial tangent to the left circle and non-negative on both disks belongs to $\text{QM}(p)_4$. The points on the left circle that are on the boundary of $\text{conv}(S)$ are of the form $a_\vartheta := (\cos(\vartheta) - 1, \sin(\vartheta))$, for some $\vartheta \in [\pi/2, 3\pi/2]$. An affine linear polynomial ℓ_ϑ defining the tangent to a_ϑ such that $\ell_\vartheta \geq 0$ on S is given by

$$\ell_\vartheta = 1 - \cos(\vartheta) - \cos(\vartheta)x_1 - \sin(\vartheta)x_2.$$

Since $\cos(\vartheta) \leq 0$ it is enough to check the equality

$$\begin{aligned}(8 - 8\cos(\vartheta)) \cdot \ell_\vartheta \quad &= p + (x_1^2 + x_2^2 - 2 + 2\cos(\vartheta))^2 + \\ &\quad + \left(2\sqrt{1 - \cos(\vartheta)}(x_2 - \sin(\vartheta))\right)^2 \\ &\quad + \left(2\sqrt{-\cos(\vartheta)}(x_1 - \cos(\vartheta) + 1)\right)^2.\end{aligned}$$

So although the fourth Lasserre relaxation equals $\operatorname{conv}\big(\mathcal{S}(\underline{p})\big)$ in this example, it still has a non-exposed face, as seen on the right in Figure 5.1.

Chapter 6

Closures and Interiors

6.1 The Closure

So far we were mostly concerned with *closed* semi-algebraic sets. It is however also interesting to consider non-closed sets. In view of the conjecture of Helton and Nie, that every convex semi-algebraic set is a spectrahedral shadow, this case should be considered as well. We will do this in the present chapter. The results are mostly published in Netzer [37]. We will however start with a converse result: the closure of a spectrahedral shadow is again a spectrahedral shadow. The result is from Gouveia and Netzer [14]. We have already used it earlier.

Proposition 6.1.1. *If $S \subseteq \mathbb{R}^n$ is a spectrahedral shadow, then so is its closure $\overline{S}$.*

Proof. By Proposition 4.1.8 we know that $(S^\circ)^\circ$ is a spectrahedral shadow. Note that this double polar lives by definition in the space of all linear polynomials defined on $\mathbb{R}[\underline{x}]_1$, which equals $\mathbb{R}^{n+2}$. If we again understand points $a \in \mathbb{R}^n$ as linear polynomials on $\mathbb{R}[\underline{x}]_1$, by the rule

$$\ell \mapsto \ell(a),$$

we find

$$(S^\circ)^\circ \cap \mathbb{R}^n = \overline{S},$$

which proves the claim. □

It is more interesting to pass from closed to non-closed sets. This is what we want to do in the following sections.

6.2 Some Helpful Results

In this section we collect some results that will be used for the main theorems in the next section. We first repeat some important notions. Let $S \subseteq \mathbb{R}^n$ be convex. The *relative interior*

$$\operatorname{relint}(S)$$

of S is the subset of S that forms the interior of S in the affine hull of S. So a point $a \in S$ belongs to $\operatorname{relint}(S)$ if and only if for all points $b \in S$ there is some $\varepsilon > 0$ such that

$$a + \varepsilon(a - b) \in S.$$

If $a \in \operatorname{relint}(S)$, then another point $b \in S$ belongs to $\operatorname{relint}(S)$ if and only if there is some $\varepsilon > 0$ such that $b + \varepsilon(b - a) \in S$. One has $S \subseteq \overline{\operatorname{relint}(S)}$.

Lemma 6.2.1. *Let $S \subseteq \mathbb{R}^n$ be a convex set and let S' be a convex subset of S which is dense in S. Then S' contains the relative interior* $\operatorname{relint}(S)$ *of S.*

Proof. Without loss of generality assume that S and therefore also S' has nonempty interior in $\mathbb{R}^n$. Now assume for contradiction that there is some $a \in \operatorname{int}(S)$ that does not belong to S'. Then by separation of disjoint convex sets, we find an affine linear polynomial $0 \neq \ell \in \mathbb{R}[\underline{x}]_1$ with $\ell(a) \leq 0$ and $\ell \geq 0$ on S'. Since S' has nonempty interior there is some $b \in S'$ with $\ell(b) > 0$. Since $S' \subseteq S$ and $a \in \operatorname{int}(S)$ we find some $\varepsilon > 0$ such that $b' := a + \varepsilon(a - b) \in S$. Since $\ell(b') < 0$ and $\ell \geq 0$ on $\overline{S'}$, this contradicts $S \subseteq \overline{S'}$. □

Corollary 6.2.2. *Let $T \subseteq \mathbb{R}^N$ be convex and let $L\colon \mathbb{R}^N \to \mathbb{R}^n$ be a linear map. Then*

$$L(\operatorname{relint}(T)) = \operatorname{relint}(L(T)).$$

Proof. The inclusion "$\subseteq$" is clear. For "$\supseteq$" notice that since $\operatorname{relint}(T)$ is convex and dense in T, $L(\operatorname{relint}(T))$ is a convex and dense subset of $L(T)$. So the claim follows from Lemma 6.2.1. □

We now turn to matrices. The next proposition will be crucial for the results in Section 6.3.

Proposition 6.2.3. *Let $M \in \mathrm{Sym}_k(\mathbb{R})$ and $N \in \mathbb{R}^{l\times k}$. Let I_l denote the identity matrix of size l. Then the following are equivalent:*

(1) there is some $\lambda \in \mathbb{R}$ such that $\left(\begin{array}{c|c} M & N^t \\ \hline N & \lambda \cdot I_l \end{array}\right) \succeq 0.$

(2) $M \succeq 0$ and $\ker M \subseteq \ker N$.

Proof. By Theorem 1 in Albert [2], (1) is equivalent to the existence of some λ such that

$$M \succeq 0,\ N = NM^\dagger M,\ \lambda \cdot I_l - NM^\dagger N^t \succeq 0,$$

where $M^\dagger$ denotes the Penrose-Moore pseudoinverse of M. By Theorem 9.17 in Ahlbrandt and Peterson [1], condition $N = NM^\dagger M$ is equivalent to $\ker M \subseteq \ker N$. Finally, one can always choose some big enough λ to insure $\lambda \cdot I_l - NM^\dagger N^t \succeq 0$, which proves the Proposition. □

6.3 Interiors

Most of the existing results on spectrahedral shadows concern closed sets. Our goal in this section is to examine non-closed sets. The following easy result states that we can always remove faces of spectrahedral shadows, and still obtain spectrahedral shadows. It does not use the results from Section 6.2 yet.

Proposition 6.3.1. *If S is a spectrahedral shadow and F is a face of S, then F and $S \setminus F$ are spectrahedral shadows.*

Proof. First assume that $S \subseteq \mathbb{R}^n$ is a spectrahedron, defined by the linear matrix polynomial $\mathcal{M}$. Then F is an exposed face of S, which means that there is an affine linear polynomial $\ell \in \mathbb{R}[\underline{x}]_1$ such that $\ell \geq 0$ on S and $\{\ell = 0\} \cap S = F$. So we have

$$F = \{a \in \mathbb{R}^n \mid \mathcal{M}(a) \succeq 0 \wedge \ell(a) = 0\}$$

and

$$S \setminus F = \left\{a \in \mathbb{R}^n \mid \mathcal{M}(a) \succeq 0 \wedge \exists\lambda \begin{pmatrix} \lambda & 1 \\ 1 & \ell(a) \end{pmatrix} \succeq 0\right\}.$$

This shows that F is even a spectrahedron and $S \setminus F$ is a spectrahedral shadow.

Now let S be a spectrahedral shadow and let $T \subseteq \mathbb{R}^N$ be a spectrahedron such that $S = L(T)$ for some linear map $L\colon \mathbb{R}^N \to \mathbb{R}^n$. Then

$$\widetilde{F} := L^{-1}(F) \cap T$$

is a face of T. We thus already know that $\widetilde{F}$ and $T \setminus \widetilde{F}$ are spectrahedral shadows. Since $\widetilde{F}$ is mapped onto F and $T \setminus \widetilde{F}$ is mapped onto $S \setminus F$ under L, both sets are also spectrahedral shadows. □

For spectrahedral shadows with only finitely many faces, i.e. for polyhedra, we thus know that the interior is again a spectrahedral shadow. But this result is true in general:

Proposition 6.3.2. *If S is a spectrahedral shadow, then* $\operatorname{relint}(S)$ *is also a spectrahedral shadow.*

Proof. First assume that $S \subseteq \mathbb{R}^n$ is a spectrahedron, defined by the matrix polynomial

$$\mathcal{M} = M_0 + x_1 M_1 + \ldots + x_n M_n.$$

Fix a point $b \in \operatorname{relint}(S)$. We know that $\operatorname{relint}(S)$ has the following description:

$$\operatorname{relint}(S) = \{a \in S \mid \exists \varepsilon > 0 \ a + \varepsilon(a-b) \in S\}.$$

For $\varepsilon > 0$ we have $\mathcal{M}(a+\varepsilon(a-b)) \succeq 0$ if and only if $\frac{1}{1+\varepsilon} \cdot \mathcal{M}(a+\varepsilon(a-b)) \succeq 0$, and

$$\begin{aligned} \frac{1}{1+\varepsilon} \cdot \mathcal{M}(a+\varepsilon(a-b)) = & \left(\frac{1}{1+\varepsilon}\right) \cdot M_0 + a_1 M_1 + \cdots + a_n M_n \\ & - \left(\frac{\varepsilon}{1+\varepsilon}\right) \cdot (b_1 M_1 + \cdots + b_n M_n). \end{aligned}$$

Making the transformation $\delta := \frac{1}{1+\varepsilon}$ and writing $N := -(b_1 M_1 + \cdots + b_n M_n)$ we find $\operatorname{relint}(S)$ to equal the following set:

$$\{a \in \mathbb{R}^n \mid \exists \delta \in (0,1) : \delta M_0 + a_1 M_1 + \cdots + a_n M_n + (1-\delta) N \succeq 0\}.$$

Since the condition $\delta \in (0,1)$ can be translated into

$$\exists \lambda \begin{pmatrix} \lambda & 1 \\ 1 & \delta \end{pmatrix} \succeq 0 \wedge \begin{pmatrix} \lambda & 1 \\ 1 & 1-\delta \end{pmatrix} \succeq 0,$$

this is clearly a realization of $\mathrm{relint}(S)$ as a spectrahedral shadow.

Now let S be an arbitrary spectrahedral shadow. Suppose $T \subseteq \mathbb{R}^N$ is a spectrahedron that maps onto S under a linear map. Then $\mathrm{relint}(T)$ maps onto $\mathrm{relint}(S)$, by Corollary 6.2.2. Since we already know that $\mathrm{relint}(T)$ is a spectrahedral shadow, this proves the claim. $\square$

Remark 6.3.3. We could also try to quantify the element b in the proof of Proposition 6.3.2, instead of only using one fixed b from $\mathrm{relint}(S)$. This would allow to be more sophisticated in removing faces of S. However, the approach from the proof doesn't seem to work then. It relies on the fact that we consider b as a fixed parameter. However, we can still prove something better, using a different method. This is our main result, Theorem 6.3.5 below.

By now we have shown that we can remove *finitely many faces* or *all faces* of codimension ≥ 1 from a spectrahedral shadow, and obtain a spectrahedral shadow. But we would also like to do something in between, for example remove a semi-arc from the boundary of the disk. With the results from the previous section we can indeed prove more.

For sets $S' \subseteq S$ we denote by $(S' \looparrowleft S)$ the set that one obtains from S by removing all faces that do not touch S':

$$(S' \looparrowleft S) := S \setminus \bigcup_{F \cap S' = \emptyset} F,$$

where F runs through all faces of S. One easily checks that this is the same as taking the union of the relative interiors of all faces of S that are touched by S', i.e.

$$(S' \looparrowleft S) = \bigcup_{F \cap S' \neq \emptyset} \mathrm{relint}(F).$$

Example 6.3.4. On the left in Figure 6.1 you see the set

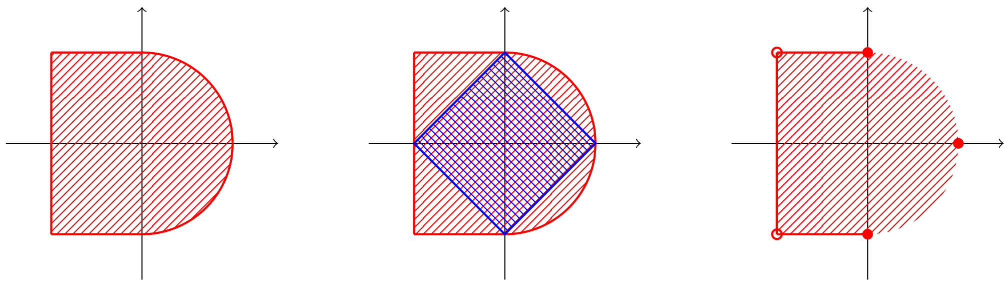
$$S = [-1, 0] \times [-1, 1] \cup \left\{ (a_1, a_2) \in \mathbb{R}^2 \mid a_1^2 + a_2^2 \leq 1 \right\}.$$

It was already examined in Example 4.3.4 and in Section 1.3. We know that it is a spectrahedral shadow. In the center of Figure 6.1 we included the subset

$$S' = \{ (a_1, a_2) \mid |a_1| + |a_2| \leq 1 \}.$$

Finally, on the right, you see the set $(S' \looparrowleft S)$. All faces of S that do not intersect S' have been removed.

Figure 6.1:



Before we state our main result, recall the following. For every point $a \in S$ there is a unique face F_a of S that contains a in its relative interior. It is the smallest face containing a. F_a consist precisely of the points $b \in S$ for which there is some $\varepsilon > 0$ such that $a + \varepsilon(a - b) \in S$.

If $S \subseteq \mathbb{R}^n$ is a spectrahedron, defined by the size k linear matrix inequality $\mathcal{M} \succeq 0$, then every face of S is of the form

$$F_U = \{a \in S \mid U \subseteq \ker \mathcal{M}(a)\}$$

for some subspace U of $\mathbb{R}^k$, and one has

$$F_a = F_{\ker \mathcal{M}(a)}$$

for all $a \in S$ (see Ramana and Goldmann [48] and Section 1.3 above).

Theorem 6.3.5. *Let $S' \subseteq S \subseteq \mathbb{R}^n$ be spectrahedral shadows. Then*

$$(S' \looparrowleft S)$$

is also a spectrahedral shadow.

Proof. First assume that S is a spectrahedron. Let $\mathcal{M}$ be a symmetric linear matrix polynomial of size k defining S. We write $\mathcal{F}(b, S)$ for the set of all faces of S containing b. For any $b \in S'$ we have

$$\begin{aligned}\bigcup_{F \in \mathcal{F}(b,S)} \operatorname{relint}(F) &= \{a \in S \mid b \in F_a\} \\ &= \{a \in \mathbb{R}^n \mid \mathcal{M}(a) \succeq 0, \ker \mathcal{M}(a) \subseteq \ker \mathcal{M}(b)\}\,.\end{aligned}$$

So by Proposition 6.2.3 we have

$$(S' \looparrowleft S) = \left\{ a \in \mathbb{R}^n \mid \exists b \in S' \ \exists \lambda \ \left(\begin{array}{c|c} \mathcal{M}(a) & \mathcal{M}(b) \\ \hline \mathcal{M}(b) & \lambda \cdot I_k \end{array} \right) \succeq 0 \right\},$$

using the second definition of $(S' \looparrowleft S)$. This proves the claim.

Now let S be an arbitrary spectrahedral shadow. So there is a linear map $L\colon \mathbb{R}^N \to \mathbb{R}^n$ and a spectrahedron $T \subseteq \mathbb{R}^N$ with $L(T) = S$. We set

$$T' := L^{-1}(S') \cap T.$$

This is clearly a spectrahedral shadow, and so we already know that $(T' \looparrowleft T)$ is a spectrahedral shadow. We thus finish the proof by showing

$$L\left((T' \looparrowleft T)\right) = (S' \looparrowleft S).$$

For "$\subseteq$" let $c \in (T' \looparrowleft T)$. If F is any face of S with $F \cap S' = \emptyset$, then clearly

$$\widetilde{F} := L^{-1}(F) \cap T$$

is a face of T with $\widetilde{F} \cap T' = \emptyset$. Since $c \in (T' \looparrowleft T)$ we thus know $c \notin \widetilde{F}$, and so $L(c) \notin F$. This proves $L(c) \in (S' \looparrowleft S)$.

For "$\supseteq$" we again use the second definition of $(S' \looparrowleft S)$. So let F be a face of S with $F \cap S' \neq \emptyset$. Then $\widetilde{F} = L^{-1}(F) \cap T$ is a face of T with $\widetilde{F} \cap T' \neq \emptyset$. So

$$\operatorname{relint}(\widetilde{F}) \subseteq (T' \looparrowleft T),$$

and by Lemma 6.2.2 we find

$$\operatorname{relint}(F) = \operatorname{relint}\left(L(\widetilde{F})\right) = L\left(\operatorname{relint}(\widetilde{F})\right) \subseteq L\left((T' \looparrowleft T)\right).$$

This finishes the proof. □

Remark 6.3.6. (0) One has $(S \looparrowleft S) = S$ and $(\emptyset \looparrowleft S) = \emptyset$ for any convex set S. Clearly $S'' \subseteq S' \subseteq S$ implies $(S'' \looparrowleft S) \subseteq (S' \looparrowleft S)$.

(1) For a point $a \in \operatorname{relint}(S)$ one has $(\{a\} \looparrowleft S) = \operatorname{relint}(S)$. So Theorem 6.3.5 generalizes Proposition 6.3.2 from above.

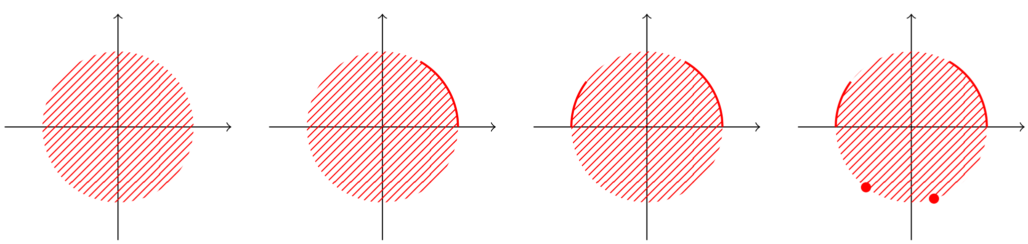
(2) $(S' \looparrowleft S)$ always contains S', and also $\operatorname{relint}(S)$ as long as $S' \neq \emptyset$.

(3) The realization of $(S' \looparrowleft S)$ as a spectrahedral shadow is explicitly given in the proof of Theorem 6.3.5. So one for example checks that it preserves rational coefficients from a realization of S' and S.

Example 6.3.7. The set on the right in Figure 6.1 is a spectrahedral shadow.

Example 6.3.8. Let D_2 be the unit disk in $\mathbb{R}^2$. We find that we can remove any arc in the boundary of D_2 (and therefore any semi-algebraic subset of the boundary) and obtain a spectrahedral shadow. See Figure 6.2 for examples. For any arc in the boundary of D_2 one simply has to provide a spectrahedral shadow in D_2 that touches the boundary of D_2 precisely in the points that do not belong to the given arc. This is always possible, as one easily checks.

Figure 6.2:



Bibliography

[1] C. D. Ahlbrandt and A. C. Peterson. *Discrete Hamiltonian systems. Difference equations, continued fractions, and Riccati equations*, vol. 16 of *Kluwer Texts in the Mathematical Sciences*. Kluwer Academic Publishers Group, Dordrecht, 1996.

[2] A. Albert. Conditions for positive and nonnegative definiteness in terms of pseudoinverses. *SIAM J. Appl. Math.*, **17**, 434–440, 1969.

[3] E. M. Alfsen. *Compact convex sets and boundary integrals*. Springer-Verlag, New York, 1971. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 57.

[4] A. Barvinok. *A course in convexity*, vol. 54 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.

[5] S. Basu, R. Pollack, and M.-F. Roy. *Algorithms in real algebraic geometry*, vol. 10 of *Algorithms and Computation in Mathematics*. Springer-Verlag, Berlin, second edn., 2006.

[6] A. Ben-Tal and A. Nemirovski. *Lectures on modern convex optimization. Analysis, algorithms, and engineering applications*. MPS/SIAM Series on Optimization. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2001.

[7] J. Bochnak, M. Coste, and M.-F. Roy. *Real algebraic geometry*, vol. 36 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*. Springer-Verlag, Berlin, 1998.

[8] P. Brändén. Obstructions to determinantal representability. *Adv. Math.*, **226 (2)**, 1202–1212, 2011.

[9] R. Brauer and H. Weyl. Spinors in n Dimensions. *Amer. J. Math.*, **57 (2)**, 425–449, 1935.

[10] L. N. Childs. Linearizing of n-ic forms and generalized Clifford algebras. *Linear and Multilinear Algebra*, **5 (4)**, 267–278, 1977/78.

[11] P. H. Diananda. On non-negative forms in real variables some or all of which are non-negative. *Proc. Cambridge Philos. Soc.*, **58**, 17–25, 1962.

[12] S. Friedland, J. W. Robbin, and J. H. Sylvester. On the crossing rule. *Comm. Pure Appl. Math.*, **37 (1)**, 19–37, 1984.

[13] D. Gondard and P. Ribenboim. Le 17e problème de Hilbert pour les matrices. *Bull. Sci. Math. (2)*, **98 (1)**, 49–56, 1974.

[14] J. Gouveia and T. Netzer. Positive polynomials and projections of spectrahedra. to appear in *SIAM J. Optim.*

[15] J. Gouveia, P. Parrilo, and R. R. Thomas. Theta bodies for polynomial ideals. *SIAM Journal on Optimization*, **20 (4)**, 2097–2118, 2010.

[16] L. Gårding. An inequality for hyperbolic polynomials. *J. Math. Mech.*, **8**, 95–965, 1959.

[17] N. Heerema. An algebra determined by a binary cubic form. *Duke Math. J.*, **21**, 423–443, 1954.

[18] J. W. Helton, I. Klep, and S. McCullough. The matricial relaxation of a linear matrix inequality. *Preprint*, 2010.

[19] J. W. Helton and J. Nie. Sufficient and necessary conditions for semidefinite representability of convex hulls and sets. *SIAM J. Optim*, **20 (2)**, 759–791, 2009.

[20] ———. Semidefinite representation of convex sets. *Math. Prog., Ser. A*, **122 (1)**, 21–62, 2010.

[21] J. W. Helton and V. Vinnikov. Linear matrix inequality representation of sets. *Comm. Pure Appl. Math.*, **60 (5)**, 654–674, 2007.

[22] D. Henrion. Detecting rigid convexity of bivariate polynomials. *Linear Algebra Appl.*, **432 (5)**, 1218–1233, 2010.

[23] ———. Semidefinite representation of convex hulls of rational varieties. *LAAS-CNRS Research Report No. 09001*, January 2009.

[24] V. A. Jakubovič. Factorization of symmetric matrix polynomials. *Dokl. Akad. Nauk SSSR*, **194**, 532–535, 1970.

[25] S. Kuhlmann, M. Marshall, and N. Schwartz. Positivity, sums of squares and the multi-dimensional moment problem. II. *Adv. Geom.*, **5 (4)**, 583–606, 2005.

[26] J. B. Lasserre. Global optimization with polynomials and the problem of moments. *SIAM J. Optim.*, **11 (3)**, 796–817 (electronic), 2000/01.

[27] ———. Convex sets with semidefinite representation. *Math. Program.*, **120 (2, Ser. A)**, 457–477, 2009.

[28] A. S. Lewis, P. A. Parrilo, and M. V. Ramana. The Lax conjecture is true. *Proc. Amer. Math. Soc.*, **133 (9)**, 2495–2499 (electronic), 2005.

[29] R. Loewy and N. Radwan. Spaces of symmetric matrices of bounded rank. *Linear Algebra Appl.*, **197/198**, 189–215, 1994.

[30] L. Lovász. On the Shannon capacity of a graph. *IEEE Trans. Inform. Theory*, **25 (1)**, 1–7, 1979.

[31] M. Marshall. Optimization of polynomial functions. *Canad. Math. Bull.*, **46 (4)**, 575–587, 2003.

[32] ———. *Positive polynomials and sums of squares*, vol. 146 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2008.

[33] R. Meshulam. On two extremal matrix problems. *Linear Algebra Appl.*, **114/115**, 261–271, 1989.

[34] K. G. Murty. *Linear Programming*. Wiley & Sons, 1983.

[35] A. Nemirovski. Advances in convex optimization: conic programming. In *International Congress of Mathematicians. Vol. I*, pp. 413–444. Eur. Math. Soc., Zürich, 2007.

[36] Y. Nesterov and A. Nemirovski. *Interior-point polynomial algorithms in convex programming*, vol. 13 of *SIAM Studies in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1994.

[37] T. Netzer. On semidefinite representations of non-closed sets. *Lin. Alg. Appl.*, **432**, 3072–3078, 2010.

[38] T. Netzer, D. Plaumann, and M. Schweighofer. Exposed faces of semidefinitely representable sets. *SIAM J. Optim.*, **20 (4)**, 1944–1955, 2010.

[39] T. Netzer, D. Plaumann, and A. Thom. Determinantal representations and the Hermite matrix. *Preprint*, 2011.

[40] T. Netzer and R. Sinn. A note on the convex hull of finitely many projections of spectrahedra. *arXiv:0908.3386v1 [math.OC]*, 2009.

[41] T. Netzer and A. Thom. Polynomials with and without determinantal representations. *Preprint*, 2011.

[42] W. Nuij. A note on hyperbolic polynomials. *Math. Scand.*, **23**, 69–72 (1969), 1968.

[43] C. J. Pappacena. Matrix pencils and a generalized Clifford algebra. *Linear Algebra Appl.*, **313 (1-3)**, 1–20, 2000.

[44] D. Plaumann, B. Sturmfels, and C. Vinzant. Computing linear matrix representations of helton-vinnikov curves. *Preprint*, 2010.

[45] V. Powers and C. Scheiderer. The moment problem for non-compact semialgebraic sets. *Adv. Geom.*, **1 (1)**, 71–88, 2001.

[46] A. Prestel and C. N. Delzell. *Positive polynomials*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2001.

[47] M. Ramana and A. Goldman. Quadratic maps with convex images. Tech. rep., Rutgers Center for Operations Research, 1995.

[48] M. Ramana and A. J. Goldman. Some geometric results in semidefinite programming. *J. Global Optim.*, **7 (1)**, 33–50, 1995.

[49] J. Renegar. Hyperbolic programs, and their derivative relaxations. *Found. Comput. Math.*, **6 (1)**, 59–79, 2006.

[50] N. Roby. Algèbres de Clifford des formes polynomes. *C. R. Acad. Sci. Paris Sér. A-B*, **268**, A484–A486, 1969.

[51] P. Rostalski and B. Sturmfels. Dualities in convex algebraic geometry. *Rendiconti di Mathematica, Serie VII*, **30**, 285–327, 2010.

[52] R. Sanyal. On the derivative cones of polyhedral cones. *Preprint*, 2011.

[53] C. Scheiderer. Convex hulls of curves of genus one. to appear in *Adv. Math.*

[54] ———. Sums of squares of regular functions on real algebraic varieties. *Trans. Amer. Math. Soc.*, **352 (3)**, 1039–1069, 2000.

[55] ———. Sums of squares on real algebraic surfaces. *Manuscripta Math.*, **119 (4)**, 395–410, 2006.

[56] ———. Weighted sums of squares in local rings and their completions, II. *Math. Z.*, **266 (1)**, 21–42, 2010.

[57] M. Schweighofer. Optimization of polynomials on compact semialgebraic sets. *SIAM J. Optim.*, **15 (3)**, 805–825 (electronic), 2005.

[58] M. Todd. Semidefinite optimization. *Acta Numerica*, **10**, 515–560, 2001.

[59] L. Vandenberghe and S. Boyd. Semidefinite programming. *SIAM Rev.*, **38 (1)**, 49–95, 1996.

[60] D. G. Wagner and Y. Wei. A criterion for the half-plane property. *Discrete Math.*, **309 (6)**, 1385–1390, 2009.

[61] H. Wolkowicz, R. Saigal, and L. Vandenberghe (editors). *Handbook of semidefinite programming. Theory, algorithms, and applications.* International Series in Operations Research & Management Science, 27. Kluwer Academic Publishers, Boston, MA, 2000.

Deutsche Zusammenfassung

Die Theorie der Spektraeder und ihrer Schatten ist ein faszinierendes Gebiet aktiver mathematischer Forschung. Das Interesse daran entstand aus der Entwicklung der semidefiniten Optimierung gegen Ende des letzten Jahrhunderts.

Semidefinite Optimierung ist eine Verallgemeinerung der linearen Optimierung. In der linearen Optimierung möchte man eine lineare Funktion unter linearen Nebenbedingungen optimieren. Optimiert wird also über einen Polyeder. Da viele Optimierungsprobleme als lineare Probleme dargestellt werden können, ist es nicht verwunderlich, dass die lineare Optimierung mindestens seit dem zweiten Weltkrieg intensiv studiert wurde. Es gibt äußerst effiziente Algorithmen, um lineare Probleme zu lösen.

In der semidefiniten Optimierung werden nun die linearen Nebenbedingungen dadurch ersetzt, dass eine Linearkombination gewisser symmetrischer oder hermitescher Matrizen M_i positiv semidefinit sein muss:

$$\mathcal{M} = I + x_1 M_1 + \cdots + x_n M_n \succeq 0$$

$\mathcal{M}$ nennt man dabei ein *lineares Matrixpolynom*. Die Größe der dabei auftretenden Matrizen M_i ist die *Größe des Matrixpolynoms.* Die Mengen, die so als zulässige Mengen entstehen, sind immer noch konvex, aber im Allgemeinen keine Polyeder mehr. Man nennt sie *Spektraeder*:

$$\mathcal{S}(\mathcal{M}) := \{a \in \mathbb{R}^n \mid \mathcal{M}(a) = I + a_1 M_1 + \cdots + a_n M_n \succeq 0\}.$$

Durch die Abschwächung der Nebenbedingungen lässt sich die semidefinite Optimierung natürlich breiter anwenden als die lineare Optimierung. Gleichzeitig gibt es immer noch sehr effiziente Lösungsalgorithmen. Dadurch wird die semidefinite Optimierung sehr interessant für Anwendungen.

Durch die Entwicklung der semidefiniten Optimierung stellt sich natürlich die Frage nach deren theoretischen Grundlagen. Eine der interessantesten Fragen ist dabei die Klassifizierung von Spektraedern. Natürlich kann es manchmal schon schwierig zu entscheiden sein, ob eine gegebene Menge ein Polyeder ist. Bei Spektraedern ist die Frage jedoch grundsätzlich schwierig. Selbst bei einer explizit in der Ebene oder im Raum gegebenen Menge ist oft nicht klar, ob es sich um einen Spektraeder handelt.

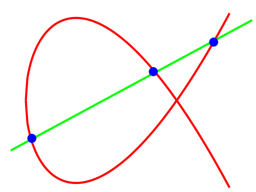
Helton und Vinnikov haben zu diesem Problem grundlegende Arbeit geleistet. Sie definieren zuerst sogenannte *starr konvexe Mengen*. Zunächst nennt man ein Polynom $p \in \mathbb{R}[x_1, \ldots, x_n]$ ein *RZ-Polynom*, falls $p(0) = 1$ gilt, und p auf allen Ursprungsgeraden nur reelle Nullstellen hat:

$$\forall a \in \mathbb{R}\colon\ p(\lambda \cdot a) = 0 \Rightarrow \lambda \in \mathbb{R}.$$

Das folgende Bild zeigt die Nullstellenmenge des Polynoms

$$p = x_1^3 - x_1^2 - x - x_2^2 + 1 \in \mathbb{R}[x_1, x_2],$$

und die Schnittpunkte mit einer Ursprungsgeraden.

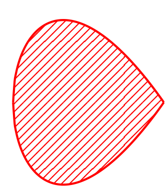


Da p vom Grad 3 ist sieht man, dass es auf jeder Ursprungsgerade nur reelle Nullstellen hat. Also ist p ein RZ-Polynom.

Die Punkte innerhalb der innersten Schale von Nullstellen von p bilden dann eine *starr konvexe Menge*:

$$\mathcal{R}(p) = \{a \in \mathbb{R}^n \mid p(\lambda \cdot a) \neq 0\ \forall \lambda \in [0, 1)\}.$$

Im folgenden Bild sieht man die starr konvexe Menge zum Polynom $p = x_1^3 - x_1^2 - x - x_2^2 + 1$:



Wenn man nun für ein lineares Matrixpolynom die Determinante

$$p = \det \mathcal{M}$$

berechnet, so stellt man fest, dass p ein RZ-Polynom ist und

$$\mathcal{S}(\mathcal{M}) = \mathcal{R}(p)$$

gilt. Also sind Spektraeder immer starr konvex. Es ist zum Beispiel leicht zu sehen dass die Menge

$$S = \{(a_1, a_2) \in \mathbb{R}^2 \mid a_1^4 + a_2^4 \leq 1\}$$

nicht starr konvex ist. Die Randkurve der Menge ist nämlich durch die Bedingung $a_1^4 + a_2^4 = 1$ definiert, und das Polynom $p = 1 - x_1^4 + x_2^4$ ist kein RZ-Polynom. Damit ist S nun auch kein Spektraeder.

Für die umgekehrte Frage, ob nämlich eine starr konvexe Menge ein Spektraeder ist, muss man also versuchen, RZ-Polynome als Determinanten von linearen Matrixpolynomen zu realisieren. Durch diese Übersetzung des ursprünglich geometrischen Problems in ein algebraisches können Helton und Vinnikov den zweidimensionalen Fall vollständig lösen:

Theorem (Helton & Vinnikov, 2007)**.** *Jedes RZ-Polynom $p \in \mathbb{R}[x_1, x_2]$ hat eine Determinantendarstellung $p = \det \mathcal{M}$. Damit ist die starr konvexe Menge $\mathcal{R}(p)$ ein Spektraeder:*

$$\mathcal{R}(p) = \mathcal{S}(\mathcal{M}).$$

In höheren Dimensionen stimmt der erste Teil des Satzes nicht mehr: es gibt RZ-Polynome ohne Determinantendarstellung. Diese Tatsache wurde zuerst von Brändén bewiesen. Im ersten Teil dieser Arbeit zeigen wir, dass sogar *fast kein* RZ-Polynom eine Determinantendarstellung hat. Wenn man mit $\mathcal{R}_{n,d}$ die Menge aller RZ-Polynome in n Variablen vom Grad höchstens d, und mit $\mathcal{D}_{n,d}$ die Menge solcher Polynome mit Determinantendarstellung bezeichnet, so kann man zeigen:

Theorem. *Wenn entweder $n \geq 3$ fixiert und d genügend groß, oder $d \geq 4$ fixiert und n genügend groß ist, so gilt*

$$\dim \mathcal{D}_{n,d} < \dim \mathcal{R}_{n,d}.$$

Das Ergebnis basiert auf Schranken für die Größe eines linearen Matrixpolynoms, die wir zuvor beweisen. Dabei wird das Kürzungsverhalten beim Berechnen der Determinante eines solchen Matrixpolynoms analysiert:

Theorem. *Jedes $p \in \mathcal{D}_{n,d}$ ist die Determinante eines linearen Matrixpolynoms der Größe nd.*

Theorem. *Falls für $p \in \mathcal{D}_{n,d}$ der Spektraeder $\mathcal{R}(p)$ einen volldimensionalen Kegel enthält, so ist p die Determinante eines linearen Matrixpolynoms der Größe d.*

Damit können wir auch einfache und explizite Gegenbeispiele konstruieren. Das RZ-Polynom

$$(x_0+1)^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2$$

etwa hat keine Determinantendarstellung. Es gibt auch solche Gegenbeispiele mit kompakter Menge $\mathcal{R}(p)$.

Ob der zweite Teil des Satzes von Helton und Vinnikov, ob nämlich jede starr konvexe Menge ein Spektraeder ist, auch in höheren Dimensionen immer stimmt, ist eine noch offene Frage. Man könnte sie etwa beweisen, indem man für eine genügend hohe Potenz des RZ-Polynoms p eine Determinantendarstellung findet. Hierzu gibt es positive und negative Ergebnisse. Brändén hat beispielsweise gezeigt, dass ein RZ-Polynom existiert, von dem keine Potenz eine Determinantendarstellung besitzt. Wir geben nun eine neue Methode an, mit der diese Frage untersucht werden kann. Dazu wird zum Polynom p eine *Hermite-Matrix* $\mathcal{H}(p)$ konstruiert. Die Einträge von $\mathcal{H}(p)$ sind Polynome in n Variablen. Das Polynom p ist nun genau dann ein RZ-Polynom, wenn $\mathcal{H}(p)$ an jedem Punkt des $\mathbb{R}^n$ positiv semidefinit ist. Wir zeigen nun:

Theorem. *Falls eine Potenz von p eine Determinantendarstellung besitzt, so ist*

$$\mathcal{H}(p) = Q^t Q$$

eine Quadratsumme von polynomiellen Matrizen $Q \in \mathrm{M}_{k\times d}(\mathbb{R}[x_1,\ldots,x_n])$.

Diese stärkere Bedingung kann numerisch sehr gut untersucht werden, wiederum durch semidefinite Optimierung. In Brändén's Beispiel erhält man beispielsweise numerisch, dass die Hermite-Matrix keine Quadratsumme ist. Auch weitere solche Beispiele lassen sich produzieren.

Wir geben weiter auch eine positive Methode an, um eine genügend hohe Potenz eines Polynoms als Determinante zu realisieren. Dazu konstruieren wir zum Polynom p eine nicht-kommutative Algebra $\mathcal{A}(p)$ mit Involution. Falls nun $\mathcal{A}(p)$ eine endlich-dimensionale Darstellung zulässt, also einen Algebra-Homomorphismus

$$\mathcal{A}(p) \to M_k(\mathbb{C}),$$

so besitzt eine Potenz von p eine Determinantendarstellung. Für den Fall eines quadratischen Polynoms p finden wir eine explizite solche Darstellung von $\mathcal{A}(p)$, und damit eine explizite Determinantendarstellung einer Potenz von p :

Theorem 6.3.9. *Für $p \in \mathcal{R}_{n,2}$ und $r = 2^{\lfloor \frac{n}{2} \rfloor - 1}$ hat p^r eine Determinantendarstellung. Die Darstellung kann explizit konstruiert werden.*

Man bekommt daraus beispielsweise die folgende Determinantendarstellung:

$$\left((x_0+1)^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2\right)^2$$

$$= \det \begin{pmatrix} 1+x_0 & x_1 + ix_3 & x_2 + ix_4 & 0 \\ x_1 - ix_3 & 1+x_0 & 0 & -x_2 - ix_4 \\ x_2 - ix_4 & 0 & 1+x_0 & x_1 + ix_3 \\ 0 & -x_2 + ix_4 & x_1 - ix_3 & 1+x_0 \end{pmatrix}.$$

Auch wenn man keine Potenz eines Polynoms p als Determinante realisieren kann, so wäre ein Darstellung eines Vielfachen pq als Determinante immer noch wünschenswert. Insbesondere wenn q in der starr konvexen Menge $\mathcal{R}(p)$ keine Nullstellen hat, ist $\mathcal{R}(p) = \mathcal{R}(pq)$ damit immer noch ein Spektraeder. Hierfür geben wir nun eine mögliche Konstruktionsmethode an, wiederum basierend auf der Hermite-Matrix $\mathcal{H}(p)$. Man beginnt mit einer Quadratsummenzerlegung von $\mathcal{H}(p)$ *mit möglichen Nennern*:

$$q^2 \cdot \mathcal{H}(p) = Q^t Q, \quad Q \in \mathrm{M}_{k\times d}(\mathbb{R}[\underline{x}]).$$

Eine solche Zerlegung existiert für jedes RZ-Polynom p. Dann betrachtet man das folgende Diagramm von freien $\mathbb{R}[\underline{x}]$-Moduln:

$$\begin{array}{ccc} \mathbb{R}[\underline{x}]^d & \xrightarrow{Q} & \mathbb{R}[\underline{x}]^k \\ \downarrow{\scriptstyle \mathcal{L}} & & \\ \mathbb{R}[\underline{x}]^d & \xrightarrow{Q} & \mathbb{R}[\underline{x}]^k \end{array}$$

Dabei ist $\mathcal{L}$ Multiplikation mit der sogenannten *Begleitmatrix* von p:

$$\mathcal{L} = \begin{pmatrix} 0 & 0 & 0 & -p_d \\ 1 & 0 & 0 & -p_{d-1} \\ 0 & \ddots & 0 & \vdots \\ 0 & \cdots & 1 & -p_1 \end{pmatrix},$$

wobei p_i der homogene Summand vom Grad i in p ist. Es ist leicht zu sehen dass $\det(I-\mathcal{L}) = p$ gilt. Leider ist $\mathcal{L}$ aber nicht symmetrisch, und die Einträge sind nicht linear. Allerdings ist $\mathcal{L}$ symmetrisch bezüglich der Bilinearform die von $q^2 \cdot \mathcal{H}(p)$ definiert wird. Gleichzeitig ist $\mathcal{L}$ ein graduierter Morphismus vom Grad 1, wenn man $\mathbb{R}[\underline{x}]^d$ mit einer geeigneten Graduierung versieht. Die Abbildung Q übersetzt die Bilinearform $q^2 \cdot \mathcal{H}(p)$ nun aber genau in die übliche Bilinearform auf $\mathbb{R}[\underline{x}]^k$, und die Graduierung auf $\mathbb{R}[\underline{x}]^d$ in die Standardgraduierung auf $\mathbb{R}[\underline{x}]^k$. Man kann also hoffen, ein symmetrisches lineares Matrixpolynom $\mathcal{M} = x_1 M_1 + \cdots + x_n M_n$ zu finden, welches obiges Diagramm kommutativ fortsetzt. Dann erhält man:

Theorem. *Sei $\mathcal{M} = x_1 M_1 + \cdots + x_n M_n$ ein symmetrisches lineares Matrixpolynom, welches $\mathcal{M}Q = Q\mathcal{L}$ erfüllt. Dann ist p ein Faktor von $\det(I - \mathcal{M})$.*

Man beachte, dass die Bedingung $\mathcal{M}Q = Q\mathcal{L}$ auf ein lineares Gleichungssystem führt, welches gewöhnlich einfach gelöst werden kann. Die Methode kann also tatsächlich zur Konstruktion von Determinantendarstellungen verwendet werden. Sie funktioniert beispielsweise immer im Fall von quadratischen Polynomen. Allerdings gibt es auch Fälle, in denen das genannte Gleichungssystem nicht lösbar ist, die Methode also versagt.

Mit einer ähnlichen Vorgehensweise kann man schließlich beweisen, dass lineare symmetrische Determinantendarstellungen *mit Nennern* für jedes RZ-Polynom existieren:

Theorem. *Sei $p \in \mathbb{R}[\underline{x}]$ ein RZ-Polynom. Dann gibt es eine symmetrische Matrix $\mathcal{M} \in \mathrm{M}_k(\mathbb{R}(\underline{x}))$ mit*

$$\mathcal{M}(\lambda \cdot a) = \lambda \cdot \mathcal{M}(a)$$

für alle $\lambda \neq 0$ und $a \in \mathbb{R}^n$ für welche $\mathcal{M}(a)$ definiert ist, so dass

$$\det(I - \mathcal{M}) = p.$$

Man beachte, dass eine solche Determinantendarstellung ein algebraisches Zertifikat für die geometrische RZ-Eigenschaft von p ist. Ein solches Zertifikat existiert also für jedes RZ-Polynom.

Im zweiten Teil der Arbeit befassen wir uns mit Projektionen von Spektraedern, sogenannten *spektraedrischen Schatten*. Obwohl ja Projektionen von

Polyedern wieder Polyeder sind, stimmt das selbe für Spektraeder im Allgemeinen nicht. Spektraedrische Schatten sind also nochmals wesentlich allgemeinere Mengen als Spektraeder. Bisher ist nicht klar, ob jede konvexe semialgebraische Menge ein solcher Schatten ist. Natürlich sind spektraedrische Schatten aus Sicht der semidefiniten Optimierung immer noch interessant. Eine Funktion kann ja einfach über den Urbild-Spektraeder optimiert werden. Dabei entstehen im Optimierungsproblem allerdings zu-sätzliche Variablen.

Wir beschreiben in der vorliegenden Arbeit zunächst die wichtigsten Ergebnisse und Konstruktionsmethoden für spektraedrische Schatten. Beispielsweise kann man zeigen, dass die konvexe Hülle zweier spektraedrischer Schatten wieder ein spektraedrischer Schatten ist.

Besonders die sogenannte *Lasserre Methode* wird genauer untersucht. Dabei beginnt man mit einer basisch abgeschlossenen semi-algebraischen Menge

$$\mathcal{S}(\underline{p}) = \{a \in \mathbb{R}^n \mid p_1(a) \geq 0, \ldots, p_m(a) \geq 0\},$$

definiert durch die Polynome $\underline{p} = (p_1, \ldots, p_m)$. Man möchte nun feststellen, ob die konvexe Hülle

$$\operatorname{conv}(\mathcal{S}(\underline{p}))$$

oder deren Abschluss ein spektraedrischer Schatten ist. Lasserre gibt dafür eine Folge von solchen Schatten an, die $\operatorname{conv}(\mathcal{S}(\underline{p}))$ von außen approximieren. Dazu betrachtet man einen *trunkierten quadratischen Modul*

$$\mathrm{QM}(\underline{p})_d := \left\{\sigma_0 + \sigma_1 p_1 + \cdots + \sigma_m p_m \mid \sigma_i \in \sum \mathbb{R}[\underline{x}]^2, \deg(\sigma_0), \deg(\sigma_i p_i) \leq d\right\}.$$

Die Bezeichnung $\sum \mathbb{R}[\underline{x}]^2$ steht für die Menge aller Quadratsummen von Polynomen. Die *d-te Lasserre Relaxierung* von $\mathcal{S}(\underline{p})$ ist nun

$$\mathcal{L}(\underline{p})_d = \left\{a \in \mathbb{R}^n \mid \ell(a) \geq 0 \text{ für alle linearen Polynome } \ell \in \mathrm{QM}(\underline{p})_d\right\}.$$

Man kann zeigen dass alle $\mathcal{L}(\underline{p})_d$ spektraedrische Schatten sind, und findet

$$\overline{\operatorname{conv}}(\mathcal{S}(\underline{p})) \subseteq \mathcal{L}(\underline{p})_{d+1} \subseteq \mathcal{L}(\underline{p})_d$$

für alle d. Besonders interessant ist nun der Fall, dass die approximierende Folge exakt ist, also ab einem gewissen Index mit $\overline{\operatorname{conv}}(\mathcal{S}(\underline{p}))$ übereinstimmt.

Wir geben die folgende geometrisch notwendige Bedingung für das Funktionieren der Methode an.

Theorem. *Sei $\mathcal{S}(\underline{p}) \subseteq \mathbb{R}^n$ konvex mit nichtleerem Inneren. Falls für ein $d \in \mathbb{N}$*

$$\mathcal{S}(\underline{p}) = \mathcal{L}(\underline{p})_d$$

gilt, so hat $\mathcal{S}(\underline{p})$ nur exponierte Seiten.

Das Ergebnis kann auch rein in der Sprache der reellen Algebra formuliert werden, als Aussage über Quadratsummendarstellungen von positiven linearen Polynomen:

Theorem. *Sei $\mathcal{S}(\underline{p})$ konvex mit nichtleerem Inneren. Angenommen es existiert ein $d \in \mathbb{N}$, so dass jedes auf $\mathcal{S}(\underline{p})$ nichtnegative lineare Polynom ℓ in $\mathrm{QM}(\underline{p})_d$ liegt. Dann hat $\mathcal{S}(\underline{p})$ nur exponierte Seiten.*

Im weiteren Verlauf der Arbeit untersuchen wir nicht-abgeschlossene Menge. Über solche Mengen gibt es bisher praktisch keine Ergebnisse. Es stellt sich aber heraus, dass viele dieser Mengen spektraedrische Schatten sind. Seien dazu zunächst

$$S' \subseteq S \subseteq \mathbb{R}^n$$

zwei konvexe Mengen. Wir definieren

$$(S' \hookleftarrow S) := S \setminus \bigcup_{F \cap S' = \emptyset} F = \bigcup_{F \cap S' \neq \emptyset} \mathrm{relint}(F).$$

Die Vereinigung läuft dabei über alle Seiten von S, und $\mathrm{relint}(F)$ steht für das relative Innere von F. Man entfernt also von S alle Seiten, die S' nicht berühren. Unser Hauptergebnis ist dann:

Theorem. *Falls S' und S spektraedrische Schatten sind, so ist auch*

$$(S' \hookleftarrow S)$$

ein spektraedrischer Schatten.

Als spezieller Fall, wenn man $S' = \{a\}$ setzt, für ein $a \in \mathrm{relint}(S)$, erhält man:

Korollar. *Wenn S ein spektraedrischer Schatten ist, so auch* $\mathrm{relint}(S)$.

Auch ein umgekehrtes Resultat kann man beweisen:

Lemma. *Falls S ein spektraedrischer Schatten ist, so auch der Abschluss $\overline{S}$.*

Die meisten Ergebnisse dieser Arbeit sind bereits publiziert. Sie entstammen den Arbeiten Gouveia und Netzer [14], Netzer [37], Netzer, Plaumann und Schweighofer [38], Netzer, Plaumann und Thom [39], Netzer und Sinn [40] sowie Netzer and Thom [41].