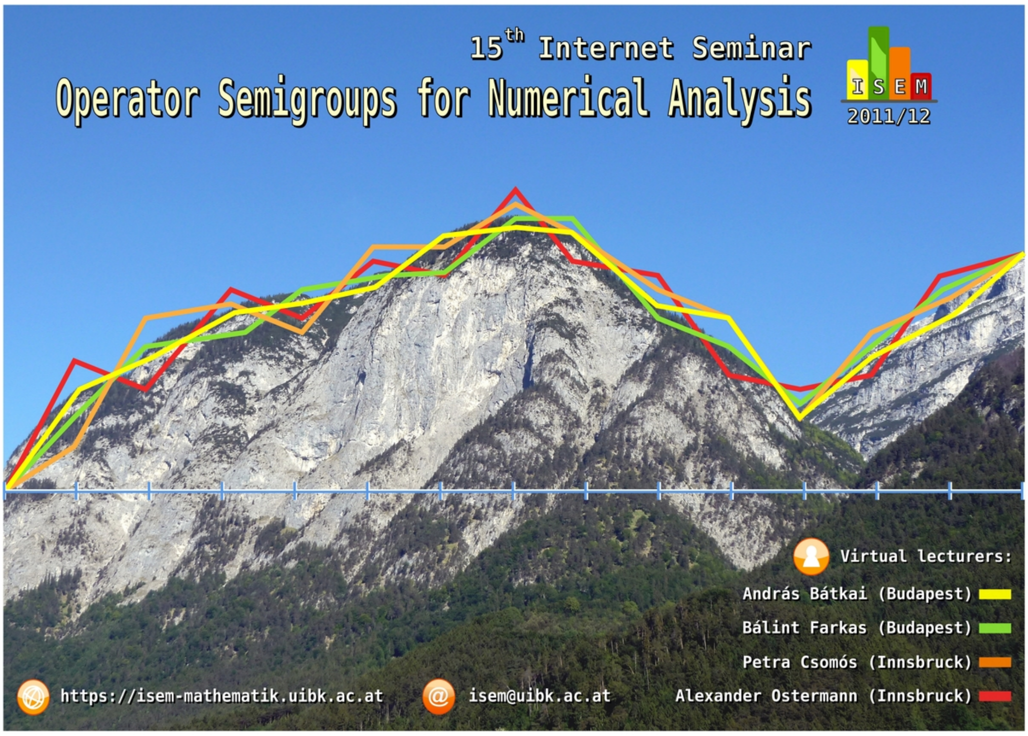
Final Workshop of the

15^{th} Internet Seminar on Evolution Equations

Operator Semigroups for Numerical Analysis



June 3 – 9, 2012

Blaubeuren, Germany

Welcome

We wish you a warm welcome to Blaubeuren, and we are looking forward to an interesting

Workshop on Operator Semigroups for Numerical Analysis.

Organised by the European consortium "International School on Evolution Equations", the annual Internet Seminars introduce master-, Ph.D. students and postdocs to varying subjects related to evolution equations. The 15$^{\text{th}}$ Internet Seminar on Evolution Equations is devoted to operator semigroup methods for numerical analysis.

So far, the course consisted of two phases.

- In *Phase 1* (October-February), a weekly lecture was provided via the ISEM website. Based on the Lax Equivalence Theorem we gave an operator theoretic and functional analytic approach to the numerical treatment of evolution equations. Our aim was to give a thorough introduction to the field, at a speed suitable for master's or Ph.D. students. The weekly lecture was accompanied by exercises, and the participants solved these problems. The lectures and the solutions to the exercises can be downloaded even now from the ISEM website

  `https://isem-mathematik.uibk.ac.at.`

- In *Phase 2* (March-May), the participants formed small international groups to work on diverse projects which complement the theory of Phase 1 and provide some applications of it.

Presently, *Phase 3* consists of the final one-week workshop at the Heinrich Fabri Institute in Blaubeuren (Germany). Here the teams will present their projects and additional lectures will be delivered by leading experts.

We would like to express our sincere thanks to all of you for following the lectures, solving the exercises, and participating in the lively discussion on the ISEM website. We hope you could also benefit a lot from elaborating the projects.

We wish you a pleasant and scientific fruitful time in Blaubeuren. If you have any questions, please do not hesitate to contact us.

András Bátkai, Petra Csomós, Bálint Farkas, and Alexander Ostermann

General Information

The workshop takes place from 3th till 9th June 2012 at the Heinrich Fabri Institute. The address is

Heinrich-Fabri-Institut

Auf dem Rucken 35
89143 Blaubeuren, Germany

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Website:	`http://www.heinrichfabriinstitut.de/`
E-mail:	`info@heinrichfabriinstitut.de`

The workshop starts in the evening of 3th June with a dinner and will end on 9th June after breakfast.

The 200 Euro conference fee includes the accommodation at the Heinrich Fabri Institute and full board (breakfast, lunch, dinner, coffee breaks).

Scientific Programme

Contributions

The talks will last 60 minutes. The project presentations are scheduled to last 90 minutes, including a brief discussion.

The conference language is English.

Equipment

The seminary room is equipped with a blackboard, an overhead projector, and a data projector. A computer with Acrobat Reader will be provided. Talks can be transferred to the conference computer through CDs or USB sticks.

Internet

There is a wireless internet access at the Heinrich Fabri Institute. The required password will be announced on the spot. We kindly ask you, however, not to use your computer during the talks.

Schedule

Monday, June 4, 2012

9.00 – 9.30	*Opening*
9.30 – 10.30	MARLIS HOCHBRUCK Exponential integrators
10.30 – 11.00	*Coffee break*
11.00 – 12.30	EXPONENTIAL SPLITTING METHODS Coord.: Katharina Schratz Yegor Dikarev, Leila Jafarian Khaled-Abad, Johannes Winckler
12.30 – 14.30	*Lunch break*
14.30 – 16.00	NONAUTONOMOUS EQUATIONS AND EVOLUTION FAMILIES Coord.: Birgit Jacob, Sven-Ake Wegner Miriam Bombieri, Felix Geyer, Andreas Geyer-Schulz, Katharina Schade
16:00 – 16.30	*Coffee break*
16.30 – 18.00	INHOMOGENEOUS AND SEMILINEAR EVOLUTION EQUATIONS Coord.: Roland Schnaubelt Martin Adler, Renata Łukasiak, David Meffert
18.30	*Dinner*

Tuesday, June 5, 2012

9.00 – 10.00	WOLFGANG ARENDT Evolution via forms and approximation
10.00 – 10.30	*Coffee break*
10.30 – 12.00	SOME POSITIVITY PRESERVING SCHEMES FOR NONLINEAR PROBLEMS Coord.: Abdelaziz Rhandi Jonathan Dehaye, Linwen Tan
12.30 – 14.30	*Lunch break*
14.30 – 15.30	COORDINATOR MEETING
15.30 – 16.30	JOB MARKET
16:00 – 16.30	*Coffee break*
16.30 – 18.00	GEOMETRIC THEORY OF SEMILINEAR PROBLEMS Coord.: Alexander Ostermann Maryna Kachanovska, Hannes Meinlschmidt, Adrian Viorel
18.30	*Dinner*

Wednesday, June 6, 2012

9.00 – 10.00	STIG LARSSON Finite element approximation of stochastic evolution PDEs
10.00 – 10.30	*Coffee break*
10.30 – 12.00	THE SEMIGROUP APPROACH TO STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY NOISE Coord.: Stig Larsson Rebekka Burkholz, Dominik Dier, Antti Koskela
12.30 – 14.30	*Lunch break*
14.30 – 18.30	*Excursion*
18.30	*Dinner*

Thursday, June 7, 2012

9.00 – 10.30	RUNGE–KUTTA DISCRETIZATIONS OF PARABOLIC PROBLEMS Coord.: Christian Lubich Sándor Kelemen, Samaneh Khodayari-Samghabadi, Balázs Kovács
10.30 – 11.10	*Coffee break*
11.00 – 12.00	CHRISTIAN LUBICH Runge–Kutta discretization of parabolic differential equations on evolving surfaces
12.30 – 14.30	*Lunch break*
14.30 – 16.00	THE STABILITY AND CONVERGENCE RESULTS OF BRENNER AND THOMÉE Coord.: Robert Haller-Dintelman Alexander Grimm, Nikita Moriakov, Felix Schwenninger
16:00 – 16.30	*Coffee break*
16.30 – 18.00	PERTURBATION THEORY OF c_0-SEMIGROUPS (THE MIYADERA THEOREM) Coord.: Jürgen Voigt Orif Ibrogimov, Matthias Lang, Chin Pin Wong, Dmitry Polyakov
19.00	*Cena Sociale*

Friday, June 8, 2012

9.00 – 10.30	Approximation results in probability theory and quantum physics Coord.: Markus Haase Björn Augner, Christoph Schuchmacher, Stefan Manuel Tomaszewski
10.30 – 11.00	*Coffee break*
11.00 – 12.30	Rational approximation of semigroups without scaling and squaring Coord.: Frank Neubrander, Koray Özer Moritz Egert, Peter Kandolf, Alina Karpikova, Leonard J. Konrad
12.30 – 14.30	*Lunch break*
14.30 – 16.00	Crank–Nicolson scheme for bounded semigroups Coord.: Hans Zwart Noémi Nagy, Austin Scirratt, Patrick Tolksdorf
16:00 – 16.30	*Coffee break*
16.30 – 18.00	Exponential quadrature Coord.: Marlis Hochbruck Ghasem Abbasi, Gábor Csörgő, Hicham El Boujaoui
18.30	*Dinner and Farewell Party*

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Talks

In alphabetical order

EVOLUTION VIA FORMS AND APPROXIMATION

Wolfgang Arendt

UNIVERSITY OF ULM, GERMANY

This talk is an introduction to form methods for evolution equations. At first we show how to come from a form to a holomorphic semigroup in Hilbert space.

Then we look more generally at non-autoonomous evolutionary problems. Forms are most appropriate for finite dimensional approximation. This leads to the Galerkin method which is of great value for theoretical results but also for numerical approximation. Finally we look at a concrete non-autonomous evolutionary problem on a network.

The interesting fact in this example is the dependance on time of the domains of the operators. In spite of this complication quite a few things can be said about the equation which go beyond well-posedness, for example one can prove asymptotic stability of the solutions.

These results were obtained most recently by the Ulm research team on non-autonous evolution, namely, Dominik Dier, Hafida Laasri, Marjeta Kramar, Delio Mugnolo and the speaker.

EXPONENTIAL INTEGRATORS

Marlis Hochbruck

KARLSRUHE INSTITUTE OF TECHNOLOGY, GERMANY

In this talk we will give an overview on the construction, analysis, implementation and application of various exponential integrators. Exponential integrators are time integration schemes, which involve the evaluation or approximation of the exponential (or related) function of a suitable matrix (e.g. the Jacobian of the differential equation). The matrix exponential of a discretized operator can be interpreted as an approximation of a semigroup, hence these methods are closely related to the material of the internet seminar.

Such methods have been proposed about 50 years ago but for a long time have been regarded as not practical. Significant advances on the approximation of the product of a matrix function with a vector during the last decade including multiple time stepping approaches have renewed the interest in these integrators. By now it has been shown that such integrators are competitive or even outperform state of the art standard methods in certain applications.

We will discuss basic ideas to construct such integrators for different applications and we will explain some convergence results for abstract partial differential equations. Moreover, we consider the approximation of matrix functions and show how these methods can be implemented in applications.

FINITE ELEMENT APPROXIMATION OF STOCHASTIC EVOLUTION PDES

Stig Larsson

CHALMERS UNIVERSITY OF TECHNOLOGY, SWEDEN

I will review our work on numerical approximation of stochastic evolution PDEs driven by noise. The equations are discretized in space by a standard finite element method and Euler's method in time. Our work includes the stochastic heat equation, wave equation and Cahn-Hilliard equation. The equations are set in an abstract framework based on operator semigroups in Hilbert space. We show strong and weak convergence of the numerical approximations.

RUNGE–KUTTA TIME DISCRETIZATION OF PARABOLIC DIFFERENTIAL EQUATIONS ON EVOLVING SURFACES

Christian Lubich

UNIVERSITY OF TÜBINGEN, GERMANY

A linear parabolic differential equation on a moving surface is first discretized in space by evolving surface finite elements and then in time by an implicit Runge–Kutta method. For algebraically stable and stiffly accurate Runge–Kutta methods, unconditional stability of the full discretization is proven and the convergence properties are analysed.

The talk is based on joint work with Gerhard Dziuk and Dhia Mansour.

Projects

In alphabetical order

APPROXIMATION RESULTS IN PROBABILITY THEORY AND QUANTUM PHYSICS

Markus Haase

BJÖRN AUGNER, CHRISTOPH SCHUCHMACHER, STEFAN MANUEL TOMASZEWSKI

The Trotter–Kato approximation theorems and the Chernoff product formula can be used to give proofs for some results in probability theory (Central Limit Theorem, Weak Law of Large Numbers) and in quantum physics (Feynman Path Formula). These ideas go back to works of Trotter [6] (in the first case) and Nelson [5] (in the second) and were developed in papers of Goldstein [3, 2, 4], inspired by [1].

The aim of this project is to give a concise presentation of these results building on Goldstein's papers and the ISEM notes as background material.

REFERENCES

[1] P. L. Butzer, W. Dickmeis, L. Hahn, and R. J. Nessel. Lax-type theorems and a unified approach to some limit theorems in probability theory with rates. *Resultate Math.*, 2(1):30–53, 1979.

[2] Jerome A. Goldstein. Corrigendum on: "Semigroup-theoretic proofs of the central limit theorem and other theorems of analysis" (Semigroup Forum **12** (1976), no. 3, 189-206). *Semigroup Forum*, 12(4):388, 1976.

[3] Jerome A. Goldstein. Semigroup-theoretic proofs of the central limit theorem and other theorems of analysis. *Semigroup Forum*, 12(3):189–206, 1976.

[4] Jerome A. Goldstein. A semigroup-theoretic proof of the law of large numbers. *Semigroup Forum*, 15(1):89–90, 1977/78.

[5] Edward Nelson. Feynman integrals and the Schrödinger equation. *J. Mathematical Phys.*, 5:332–343, 1964.

[6] H. F. Trotter. An elementary proof of the central limit theorem. *Arch. Math.*, 10:226–234, 1959.

THE STABILITY AND CONVERGENCE RESULTS OF BRENNER AND THOMÉE

Robert Haller-Dintelmann

Alexander Grimm, Nikita Moriakov, Felix Schwenninger

This project comes back to the promise of the virtual lecturers from Section 14.1 in the Isem lecture notes, that we will hear more on the stability and convergence theorems of Brenner and Thomée in the project phase. It invites you to dwelve into the proof of these two theorems.
The proof of both theorems is mainly based on the Hille-Phillips functional calculus. We saw in the course that functional calculi are a powerful tool to get stability and convergence results for rational approximation schemes and this was illustrated using the Dunford calculus for analytic semigroups in Section 13, which is a very natural choice. However, there are many interesting semigroups that are not analytic, such as shift semigroups or all semigroups that are actually groups.
That is where the Hille-Phillips calculus comes in. As was already pointed out in Section 14.1, the basic idea is to write a holomorphic function F as the Laplace transform of a bounded Borel measure μ on $[0,\infty)$, i.e.,

$$F(z) = \int_0^\infty \mathrm{e}^{sz} \mathrm{d}\mu(s) \quad (\Re z \le 0)$$

and then to substitute the semigroup e^{sA} for the term e^{sz}

$$F(A) = \int_0^\infty \mathrm{e}^{sA} \mathrm{d}\mu(s)$$

and to hope for the best, i.e. that the resulting integral makes sense and that this procedure gives rise to a useful functional calculus.
So, for the project there will be two main tasks: First, to build up and explain the Hille-Phillips functional calculus and then to understand and present the proofs of Theorems 14.1 and 14.2 of the lecture notes.
The project will be based on the original article of Brenner and Thomée [1] for their results and on the book of Hille and Phillips [2], as well as the PhD thesis of Mihály Kovács [3] concerning the Hille-Phillips calculus.

REFERENCES

[1] P. Brenner, V. Thomée: On rational approximations of semigroups, *SIAM Journal on Numerical Analysis* **16** (1979), no. 4, 683–694.

[2] E. Hille, R.S. Phillips: *Functional analysis and semigroups*, Colloquium Publications, American Mathematical Society (AMS), 1957.

[3] M. Kovács: *On qualitative properties and convergence of time-discretization methods for semigroups*, PhD thesis, Louisiana State University, 2004.

EXPONENTIAL QUADRATURE

Marlis Hochbruck

Ghasem Abbasi, Gábor Csörgő, Hicham El Boujaoui

The aim of this project is to study the numerical approximation to solutions of linear abstract differential equations

$$u'(t) + Au(t) = f(t), \qquad u(t_0+) = u_0$$

on a Banach space X by exponential quadrature formulas.
To define such quadrature formulas we choose non-confluent collocation nodes $c_1, \ldots, c_s$ and define approximations $u_n \approx u(t_n)$, where $t_n = t_0 + nh$, $n = 0, 1, \ldots$ via

$$u_{n+1} = \mathrm{e}^{-hA} u_n + h \sum_{i=1}^{s} b_i(-hA) f(t_n + c_i h)$$

with weights

$$b_i(-hA) = \frac{1}{h} \int_0^h \mathrm{e}^{-(h-\tau)A} \ell_i(\tau)\, \mathrm{d}\tau.$$

Here, ℓ_j is the Lagrange interpolation polynomial

$$\ell_j(\tau) = \prod_{m \neq j} \frac{\tau/h - c_m}{c_j - c_m}.$$

The project involves

- construction of exponential quadrature formulas
- convergence analysis in different Banach spaces (e.g. in L^p) and with different boundary conditions
- numerical experiments (using Matlab or any other programming language)

REFERENCES

[1] . Hochbruck, A. Ostermann: Exponential Runge–Kutta methods for parabolic problems, *Appl. Numer. Math.*, vol. 53, no. 2-4, pp. 323-339 (2005)
`http://authors.elsevier.com/sd/article/S0168927404001400`

NON-AUTONOMOUS EQUATIONS AND EVOLUTION FAMILIES

Birgit Jacob and Sven-Ake Wegner

MIRIAM BOMBIERI, FELIX GEYER, ANDREAS GEYER-SCHULZ, KATHARINA SCHADE

In this project we study differential equations with time-dependent coefficients, i.e., a non-autonomous evolution equations of the form

$$\begin{aligned} \frac{d}{dt}u(t) &= A(t)u(t), \quad t \geq s \in \mathbb{R}, \\ u(s) &= u_0, \end{aligned} \tag{1}$$

on a Banach space X. If $A(t) \equiv A$, then the solution of (1) is given by a strongly continuous semigroup $(T(t))_{t\geq 0}$. In the general situation the semigroup is replaced by a strongly continuous evolution family $(U(t,s))_{t\geq s}$; this notion we briefly met in Chapter 14.2. A family $(U(t,s))_{t\geq s}$ of linear, bounded operators on a Banach space X is called a *(strongly continuous) evolution family* if

1. $U(t,r)U(r,s) = U(t,s)$, $U(t,t) = I$ hold for all $s \leq r \leq t \in \mathbb{R}$,
2. The mapping $(t,s) \mapsto U(t,s)$ from $\{(\tau,\sigma) \in \mathbb{R}^2 \mid \tau \geq \sigma\}$ to $L(X)$ is strongly continuous.

We say that $(U(t,s))_{t\geq s}$ solves the Cauchy problem (1) if there exist dense subspaces Y_s, $s \in \mathbb{R}$, of X such that the function $t \mapsto U(t,s)x$ is a solution of the Cauchy problem (1) for $s \in \mathbb{R}$ and $x \in Y_s$. Clearly, a semigroup $(T(t))_{t\geq 0}$ defines an evolution family by $U(t,s) := T(t-s)$.
In the ISEM lecture notes, several characterizations of solvability of the autonomous Cauchy problem in terms of the operator A are given. Unfortunately, there is no analogous result in the non-autonomous situation. In this project we will develop several sufficient conditions for solvability of the Cauchy problem (1) in terms of the operators $A(t)$.

REFERENCES

[1] K.-J. Engel and R. Nagel. One-parameter semigroups for linear evolution equations. Graduate Texts in Mathematics, 194. Springer-Verlag, New York, 2000.

[2] A. Pazy. Semigroups of linear operators and applications to partial differential equations. Applied Mathematical Sciences, 44. Springer-Verlag, New York, 1983

THE SEMIGROUP APPROACH TO STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS DRIVEN BY NOISE

Stig Larsson

REBEKKA BURKHOLZ, DOMINIK DIER, ANTTI KOSKELA

The stochastic wave equation driven by additive noise,

$$\begin{aligned} &\mathrm{d}\dot{u} - \Delta u \, \mathrm{d}t = f(u) \, \mathrm{d}t + \mathrm{d}W && \text{in } \mathscr{D} \times (0,\infty), \\ &u = 0 && \text{in } \partial\mathscr{D} \times (0,\infty), \\ &u(\cdot,0) = u_0, \quad \dot{u}(\cdot,0) = v_0 && \text{in } \mathscr{D}, \end{aligned}$$

can be given a rigorous formulation

$$X(t) = \mathrm{e}^{-tA} X_0 + \int_0^t \mathrm{e}^{-(t-s)A} B f(X_1(s)) \, \mathrm{d}s + \int_0^t \mathrm{e}^{-(t-s)A} B \, \mathrm{d}W(s), \tag{2}$$

where

$$A = \begin{bmatrix} 0 & -I \\ \Lambda & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} u \\ \dot{u} \end{bmatrix}, \quad X_0 = \begin{bmatrix} X_{0,1} \\ X_{0,2} \end{bmatrix} = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}.$$

The article [2] provides a so-called weak convergence analysis for finite element approximations of linear equations of this kind ($f = 0$).
The aim of the project is to extend the analysis in [2] for the linear wave equation to the semilinear equation (7). Such analysis was done earlier for the semilinear Schrödinger equation in [1] and it uses the fact that the operator family $\{\mathrm{e}^{-tA}\}$ is a group in order to re-write the equation to a form which is easier to analyze.

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[1] A. de Bouard and A. Debussche, *Weak and strong order of convergence of a semidiscrete scheme for the stochastic nonlinear Schrödinger equation*, Appl. Math. Optim. **54** (2006), 369–399.

[2] M. Kovács, S. Larsson, and F. Lindgren, *Weak convergence of finite element approximations of linear stochastic evolution equations with additive noise*, BIT Numer. Math. (2011). [doi:10.1007/s10543-011-0344-2]

RUNGE–KUTTA DISCRETIZATIONS OF PARABOLIC PROBLEMS

Christian Lubich

SÁNDOR KELEMEN, SAMANEH KHODAYARI-SAMGHABADI, BALÁZS KOVÁCS

The present project basically relies on the two papers given in the References.

[1] ABSTRACT: We study the approximation properties of Runge–Kutta time discretizations of linear and semilinear parabolic equations, including incompressible Navier–Stokes equations. We derive asymptotically sharp error bounds and relate the temporal order of convergence, which is generally noninteger, to spatial regularity and the type of boundary conditions. The analysis relies on an interpretation of Runge-Kutta methods as convolution quadratures. In a different context, these can be used as efficient computational methods for the approximation of convolution integrals and integral equations. They use the Laplace transform of the convolution kernel via a discrete operational calculus.

[2] ABSTRACT: We study the convergence properties of implicit Runge-Kutta methods applied to time discretization of parabolic equations with time- or solution-dependent operator. Error bounds are derived in the energy norm. The convergence analysis uses two different approaches. The first, technically simpler approach relies on energy estimates and requires algebraic stability of the Runge–Kutta method. The second one is based on estimates for linear time-invariant equations and uses Fourier and perturbation techniques. It applies to $A(\Theta)$-stable Runge–Kutta methods and yields the precise temporal order of convergence. This order is noninteger in general and depends on the type of boundary conditions.

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[2] Ch. Lubich, A. Ostermann, Runge-Kutta approximation of quasilinear parabolic equations. *Math. Comp.* **64**, 601-627 (1995). DOI: 10.1090/S0025-5718-1995-1284670-0

RATIONAL APPROXIMATIONS OF SEMIGROUPS WITHOUT SCALING AND SQUARING

Frank Neubrander, Koray Özer

MORITZ EGERT, PETER KANDOLF, ALINA KARPIKOVA, LEONARD J. KONRAD

In this project we will discuss how to find for all $q \geq 1$ distinct complex numbers b_i and λ_i with $1 \leq i \leq q$ and $\mathrm{Re}(\lambda_i) > 0$ such that for any generator $(A, D(A))$ of a bounded, strongly continuous semigroup $T(t)$ on Banach space X with resolvent $R(\lambda, A) := (\lambda I - A)^{-1}$ the expression

$$\tfrac{b_1}{t} R\big(\tfrac{\lambda_1}{t}, A\big) + \tfrac{b_2}{t} R\big(\tfrac{\lambda_2}{t}, A\big) + \cdots + \tfrac{b_q}{t} R\big(\tfrac{\lambda_q}{t}, A\big)$$

provides an excellent approximation of the semigroup $T(t)$ on $D(A^{2q-1})$.

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GEOMETRIC THEORY OF SEMILINEAR PROBLEMS

Alexander Ostermann

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This project is concerned with the geometric theory of semilinear parabolic equations

$$u'(t) = Au(t) + g(u(t)) \tag{3}$$

and their numerical discretisations. Geometric theory is concerned with the qualitative behaviour of solutions, the geometry of the flow and stability questions. A good introduction into this field is the book by Dan Henry [2]. The simplest objects to be studied are asymptotically stable equilibria of (3). Such a study was partially carried out in our last ISEM lecture.

A hyperbolic equilibrium (saddle-point) is more involved as it possesses in its neighbourhood a stable and an unstable invariant manifold which generalise the concepts of stable and unstable subspaces for the linear problem. Numerical discretisations posses discrete counterparts thereof. The largest part of the project is concerned with the construction of these invariant sets.

Possible extensions cover periodic orbits (see [1] and [4]), and Hopf bifurcations (see [3]). The latter require the construction of an invariant centre manifold in which the bifurcation takes place.

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SOME POSITIVITY PRESERVING SCHEMES FOR SEMILINEAR PROBLEMS

Abdelaziz Rhandi

JONATHAN DEHAYE, IMRE FEKETE, LINWEN TAN

The aim of this project is to study the convergence and positivity properties of second-order exponential Runge–Kutta and Strang splitting methods applied to inhomogeneous and semilinear parabolic problems.

It is known that in many applications, such as e.g. population dynamics, mathematical finance, reaction kinetic..., the positivity is an important feature. Looking for numerical schemes that preserve positivity and study their convergence is not a trivial task. However, it was shown by Bolley and Crouzeix [1] that the order of an unconditionally positive Runge–Kutta method for an inhomogeneous linear parabolic problem can not exceed one.

As we will see in this project, the use of exponential integrators permit to preserve positivity and improve the convergence. The project is divided into two parts. The first one deals with the second-order exponential Runge-Kutta method applied to the inhomogeneous Cauchy problem

$$u'(t) = Au(t) + f(t), \quad u(0) = u_0. \tag{4}$$

We propose to show that the second-order exponential Runge-Kutta method preserves positivity in (4).

In the second part we are interested in applying the Strang splitting to the semilinear problem

$$u'(t) = Au(t) + f(u(t)), \quad u(0) = u_0. \tag{5}$$

Here we study consistency and convergence of the Strang splitting applied to (5). Finally, we deduce that the Strang splitting preserves positivity. Here A with domain $D(A)$ generates a positive C_0-semigroup on a Banach lattice X and f satisfies appropriate assumptions.

The first part is mainly contained in [4] and the second one in [2]. For the theory of positive semigroups we refer to [5] and for problems of type (4) and (5) we refer to [3].

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INHOMOGENEOUS AND SEMILINEAR EVOLUTION EQUATIONS

Roland Schnaubelt

MARTIN ADLER, RENATA LUKASIAK, DAVID MEFFERT

In the Internet Seminar we have treated linear Cauchy problems governed by a generator A on a Banach space X. If one adds to such a system an external 'force' (or control) $f \in C(\mathbb{R}_+, X)$, then one obtains the inhomogeneous evolution equation

$$u'(t) = Au(t) + f(t), \quad t \geq 0, \qquad u(0) = u_0.$$

If this problem has a classical solution u in C^1 sense, it is easy to see that it is given by Duhamel's formula

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)\,ds, \qquad t \geq 0,$$

where $T(\cdot)$ is generated by A. One can define this *mild solution* u for any $f \in L^1(\mathbb{R}_+, X)$, but then u does not need to be differentiable. The first aim of the project is to give conditions ensuring that the mild solution is in fact a classical one.

Many problems in the sciences are nonlinear, leading to a lot of new and interesting challenges. Here we restrict ourselves to *semilinear* problems which can be treated based on results about inhomogeneous evolution linear equations. Given a generator A on X and a locally Lipschitz map $F : X \to X$ we consider

$$u'(t) = Au(t) + F(u(t)), \quad t \geq 0, \qquad u(0) = u_0.$$

As a an example, think of a reaction diffusion equation given by, say, $A = d^2/dx^2$ with boundary conditions and $F(v) = v(1-v)$ on $X = C([0,1])$. In view of the above remarks, the solution of the semilinear problem should satisfy the fixed point problem

$$u(t) = T(t)u_0 + \int_0^t T(t-s)F(u(s))\,ds, \qquad t \geq 0,$$

and this is actually the starting point to construct a (unique) solution.

The project is based on Sections 4.2 and 6.1 of [1], where may simplify a few points and add more material concerning examples.

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EXPONENTIAL SPLITTING METHODS FOR DIMENSION SPLITTING

Katharina Schratz

YEGOR DIKAREV, LEILA JAFARIAN KHALED-ABAD, JOHANNES WINCKLER

PDEs are made by God, the boundary conditions by the Devil! [1]

This project is concerned with the convergence order of splitting methods applied as a numerical time integration method to partial differential equations, where

$$\begin{aligned} &\partial_t w(t,x,y) = \mathscr{L}(\partial_x,\partial_y)w(t,x,y), \quad (x,y)\in\Omega=(0,1)^2, t\in]0,T] \\ &w(0,x,y)=w_0(x,y) \\ &w(t,\cdot,\cdot)|_{\partial\Omega}=0 \qquad \text{for all } t\in[0,T] \end{aligned} \tag{6}$$

with a strongly elliptic differential operator $\mathscr{L}(\partial_x,\partial_y)=\partial_x\big(a(x,y)\partial_x\big)+\partial_y\big(b(x,y)\partial_y\big)$ and smooth, positive coefficients a,b. The splitting ansatz is the so-called dimension-splitting, where the differential operator $\mathscr{L}(\partial_x,\partial_y)$ is split along its dimensions, i.e.

$$\begin{aligned} &\mathscr{L}(\partial_x,\partial_y)=\mathscr{A}(\partial_x)+\mathscr{B}(\partial_y) \text{ with} \\ &\mathscr{A}(\partial_x)=\partial_x\big(a(x,y)\partial_x\big), \quad \mathscr{B}(\partial_y)=\partial_y\big(b(x,y)\partial_y\big). \end{aligned}$$

Full-order convergence of resolvent splitting methods applied to (6) was already discussed in the lecture notes, see Section 11.1. In this project we will analyze exponential splitting methods applied to our model problem (6). In particular we will discuss the convergence order of the exponential Lie and Strang splitting.

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PERTURBATION THEORY OF C_0-SEMIGROUPS (THE MIYADERA THEOREM)

Jürgen Voigt

ORIF IBROGIMOV, MATTHIAS LANG, CHIN PIN WONG, DMITRY POLYAKOV

The objects of this project are the Miyadera perturbation theorem and applications. If T is a C_0-semigoups with generator A, and B is an operator then (along with suitable technical conditions) the condition that

$$\int_0^\alpha \|BT(t)x\| \, dt \leq \gamma \|x\|$$

for suitable $\alpha > 0$, $\gamma < 1$ and all $x \in D(A)$ implies that $A+B$ is a generator. One part of the project is to understand the proof of this theorem.

The application to 'Schrödinger semigroups' (alias heat equation with absorption) yields the relation between Miyadera perturbations and the 'Kato class' of potentials. Another application of interest is the 'substochastic perturbation' of substochastic semigroups on L_1-spaces, a general version of Kolmogorov's differential equations. A third application could be the perturbation theory of delay equations, but I do not intend to include this topic in the project.

For the Miyadera perturbation theorem I refer to [4], [5], [7], [2, III.3.c], but I suggest to follow the presentation in [11, Section 3]. For the application to Schrödinger semigroups I refer to [8] (and possibly [10]). For the application to substochastic semigroups on L_1-spaces I refer to [3], [9] (and to [6] for a generalisation). A standard reference for the application to delay equations is [1].

The papers by Kato and Miyadera as well as my papers quoted below can be obtained under `http://www.math.tu-dresden.de/~voigt/isem11/proj-mpt`.

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CRANK–NICOLSON SCHEME FOR BOUNDED SEMIGROUPS

Hans Zwart

NOÉMI NAGY, AUSTIN SCIRRATT, PATRICK TOLKSDORF

In the lecture notes we have encountered the Crank-Nicolson scheme (or method) at several occasions. This scheme replaces the differential equation

$$\dot{x}(t) = Ax(t), \qquad t \geq 0, \quad x(0) = x_0 \tag{7}$$

by the difference equation

$$x_d(n+1) = \left(I + \tfrac{hA}{2}\right)\left(I - \tfrac{hA}{2}\right)^{-1} x_d(n), \quad n \in \mathbb{N}, \quad x_d(0) = x_0. \tag{8}$$

In Theorem 13.12 it is shown if A generates a bounded analytic semigroup, then $\|A_d^n\|$ is uniformly bounded, where $A_d = (I + \frac{hA}{2})(I - \frac{hA}{2})^{-1}$.

In this project we want to investigate this property when A is just the infinitesimal generator of a bounded C_0-semigroup. Hence not necessarily analytic. It turns out that the estimate

$$\|A_d^n\| \leq M\sqrt{n}$$

is the best estimate possible for general Banach spaces, but for Hilbert spaces we can get uniform boundedness for several cases:

- A generates a contraction semigroup,
- A generates an analytic semigroup,
- A and A^{-1} generate a bounded semigroup.

The aim of this project is understand these results and to apply it to some p.d.e.'s. A possible extension is to look at the best estimates if A is a matrix. These estimates will depend on the size of the matrix.

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