Lecture 7

Complex Powers of Closed Operators

Having finished with the theoretical questions on well-posedness of evolution equations, we now turn our attention on technical matters which will be extremely important in proving convergence rates for various discretisation procedures. Note that in Lectures 3 and 4 the existence of Banach space Y invariant under some given semigroup T was of enormous importance. Our aim in this lecture is to arm you with important examples for such spaces. As a warm up, let us first summarise the results of Exercise 4.1.

Proposition 7.1. *Let A be the generator of a semigroup of type (M,ω) in the Banach space X, and consider the space $X_n = D(A^n)$ with the graph norm which we denote by $\|\cdot\|_{A^n}$.*

a) For $n \in \mathbb{N}$ and $f \in D(A^n)$ define $|||f|||_n := \|f\| + \|Af\| + \cdots + \|A^n f\|$. Then $|||\cdot|||_n$ and $\|\cdot\|_{A^n}$ are equivalent norms.

b) The spaces X_n are Banach spaces and are invariant under the semigroup T. If we set $T_n(t) := T(t)|_{X_n}$, then T_n is a semigroup of type (M,ω) on X_n.

For applications these spaces are quite often too small and some intermediate spaces are needed. The purpose of this lecture is to find some possible candidates for such invariant subspaces that fit well in the scale of $D(A^n)$, $n = 1, 2, \ldots$.

To motivate this a bit further, let us consider the next example:

Example 7.2. Recall from Lecture 1 the multiplication operator M on ℓ^2 by the sequence $-n^2$, which corresponds to the Dirichlet Laplacian on $[0,\pi]$ after *diagonalisation* (more precisely after applying the spectral theorem for selfadjoint operators)

$$D(M) = \{(x_n) \in \ell^2 : (n^2 x_n) \in \ell^2\} \quad \text{and} \quad M(x_n) = (-n^2 x_n).$$

For $\alpha \geq 0$ define

$$D((-M)^\alpha) = \{(x_n) \in \ell^2 : (n^{2\alpha} x_n) \in \ell^2\} \quad \text{and} \quad (-M)^\alpha(x_n) = (n^{2\alpha} x_n).$$

(The minus sign here is only a matter of convention.) It is not hard to see that $(-M)^\alpha$ is a closed operator, hence $D((-M)^\alpha)$ is a Banach space with the graph norm. Equally easy is to see that $(-M)^k$ is indeed the k^{th} power of $(-M)$ for $k \in \mathbb{N}$, and that the semigroup T defined by

$$T(t)(x_n) = (\mathrm{e}^{-n^2 t} x_n) \in \ell^2$$

leaves this space invariant (much more(!) is true). Hence the spaces $D((-M)^\alpha)$ fulfill the requirements formulated above.

Thus we set out for the quest for fractional powers of closed operators. For the purposes of this lecture we shall leave semigroups (almost completely) behind, and develop some beautiful operator theoretic notions.

7.1 Complex powers with negative real part

We want to define complex powers of operators A, i.e., we want to plug in A into the function $F(x) = x^z$ where $z \in \mathbb{C}$ is fixed. This means that we want to develop a *functional calculus* for this particular function F and for some reasonable class of operators. To be able to do that we shall need the complex power functions defined on the complex plane. Let $\log : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ be the principal branch of the logarithm, i.e., $\log(\lambda) = \log|\lambda| + \mathrm{i}\arg(\lambda)$, where we have fixed the function arg with values in $(-\pi, \pi)$. Since log is holomorphic, we can define the holomorphic function $\lambda \mapsto \lambda^z = \mathrm{e}^{z\log(\lambda)}$ on $\mathbb{C} \setminus (-\infty, 0]$ for any given $z \in \mathbb{C}$. Now the basic idea comes from Cauchy's integral theorem for this particular situation:

$$a^z = \oint \frac{\lambda^z}{\lambda - a}\,\mathrm{d}\lambda$$

where we integrate along a curve that passes around $a \notin (-\infty, 0]$ in the positive direction and avoids the negative real axis. Therefore, by analogy, or motivated by multiplication operators (cf. Exercise 2) we have to give meaning to expressions like

$$\oint \lambda^z R(\lambda, A)\,\mathrm{d}\lambda.$$

Of course the curve that we are integrating over has to lie in the resolvent set of A and pass around the spectrum of $\sigma(A)$ in the positive direction. Two difficulties arise here immediately: the spectrum may be unbounded, hence the integration curve has to be unbounded (and anyway the term "passing around" does not make sense any more), and convergence issues for the integral have to be taken care of. This section includes a fair amount of technicalities, but the single idea has been explained above. The next assumption tackles both mentioned difficulties as we shall shortly see.

Assumption 7.3. Suppose for $A : D(A) \to X$ one has $(-\infty, 0] \subseteq \rho(A)$ and

$$\|R(\lambda, A)\| \leq \frac{M}{1 + |\lambda|} \quad \text{for all } \lambda \leq 0 \text{ and some } M \geq 0.$$

All operators[1] A occurring in this section will be assumed to have the property above. The next is an important example for such operators, leading back for a moment to semigroups.

Example 7.4. If A generates a strongly continuous semigroup of type (M', ω) with $\omega < 0$, then, as consequence of (2.2) in Proposition 2.26, we see that for $\lambda > 0$

$$\|R(\lambda, A)\| \leq \frac{M}{\lambda - \omega} \leq \frac{M'}{\lambda + 1}.$$

Hence $-A$ satisfies the above estimate in Assumption 7.3 for some M'.

The next fundamental result shows that although only $(-\infty, 0] \subseteq \rho(A)$ was assumed, one gains a sector around the negative real axis, where the resolvent can be estimated satisfactorily well.

[1] Some authors use the names *sectorial operator* or *positive operator* for objects having this property. We decided not to give them a name.

Proposition 7.5. *Suppose A is as in Assumption 7.3. Then there is $\theta_0 \in (\frac{\pi}{2}, \pi)$ and $r_0 > 0$ such that the set*

$$\Lambda := \big\{z \in \mathbb{C} : |\arg(z)| \in (\theta_0, \pi]\big\} \cup \big\{z \in \mathbb{C} : |z| \leq r_0\big\} \subseteq \rho(A)$$

belongs to the resolvent set of A. Moreover, there is $M_0 \geq 0$ so that for every $\lambda \in \Lambda$ one has

$$\|R(\lambda, A)\| \leq \frac{M_0}{1 + |\lambda|}. \tag{7.1}$$

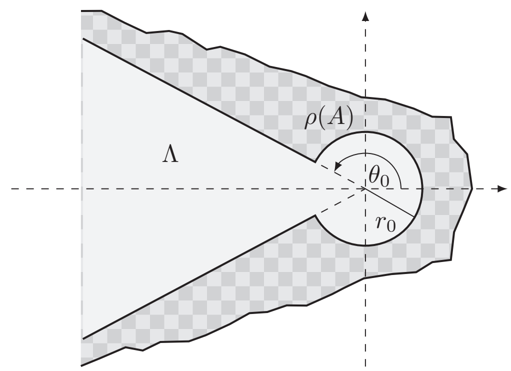

Figure 7.1: The resolvent set of A and the set Λ

Proof. First of all note that for some $r_0 > 0$ the closed ball $\overline{\mathrm{B}}(0, r_0)$ is contained in $\rho(A)$, since $\rho(A)$ is open. So on this ball the resolvent is bounded. On the other hand, we have $\mu \in \rho(A)$ and

$$R(\mu, A) = \sum_{k=0}^{\infty} (\lambda - \mu)^k R(\lambda, A)^{k+1}$$

whenever $|\mu - \lambda| < \|R(\lambda, A)\|^{-1}$, i.e., the open ball $\mathrm{B}(\lambda, -\frac{\lambda}{M})$ is contained in $\rho(A)$. From this the first assertion follows for $\theta_0 = \pi - \arctan(\frac{1}{M})$. If $|\arg \lambda| \in (\theta_0, \pi]$, then

$$\begin{aligned}\|R(\mu, A)\| &\leq \sum_{k=0}^{\infty} |\operatorname{Re}\mu - \mu|^k \frac{M^k}{(1 + |\operatorname{Re}\mu|)^{k+1}} = \sum_{k=0}^{\infty} |\operatorname{Im}\mu|^k \frac{M^k}{(1 + |\operatorname{Re}\mu|)^{k+1}} \\ &\leq \frac{M_1}{1 + |\operatorname{Re}\mu|} \leq \frac{M_0}{1 + |\mu|}.\end{aligned}$$

□

Remark 7.6. The next two estimates will be crucial for proving convergence of some integrals and for estimating them:

1. By the proposition above we have

$$\|R(\lambda, A)\| \leq \frac{M_0}{|\lambda|} \qquad \text{for all } \lambda \in \Lambda,\ |\lambda| > r_0 > 0,$$

   and

$$\|R(\lambda, A)\| \leq M_1 \qquad \text{for all } \lambda \in \Lambda,\ |\lambda| \leq r_0.$$

2. For $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ we have
$$\left|\lambda^z\right| = |\lambda|^{\operatorname{Re} z} \mathrm{e}^{-\operatorname{Im} z \cdot \arg(\lambda)} \leq |\lambda|^{\operatorname{Re} z} \mathrm{e}^{\pi |\operatorname{Im} z|} = M_2 |\lambda|^{\operatorname{Re} z}$$
for every fixed z. In particular for $\operatorname{Re} z < 0$, we have a decay as $|\lambda| \to \infty$.

We shall often use these estimates without further mentioning. Next we turn our attention to integration paths. To abbreviate a little we shall call a piecewise continuously differentiable path **admissible** if it belongs to Λ and goes from $\infty \mathrm{e}^{\mathrm{i}\theta}$ to $\infty \mathrm{e}^{-\mathrm{i}\theta}$ for some $\theta \in (\theta_0, \pi)$. Important examples for admissible curves are given by the following parametrisations:

Example 7.7. let $\theta \in (\theta_0, \pi)$ and let $\gamma_1(s) = s\mathrm{e}^{\mathrm{i}\theta} + a$ and let $\gamma_2(s) = s\mathrm{e}^{-\mathrm{i}\theta} + a$, $s \in [0, \infty)$. For $a > 0$ sufficiently small the curve $\gamma = -\gamma_1 + \gamma_2$ is admissible.

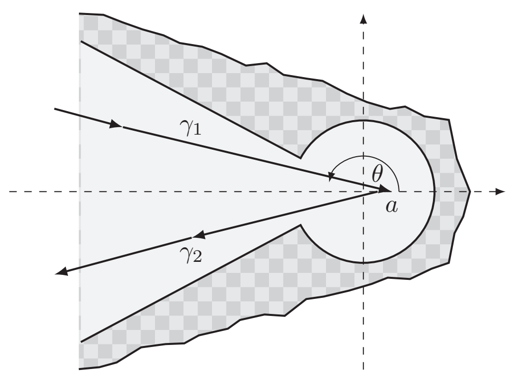

Figure 7.2: An admissible curve γ

Here is the first result giving meaning to the expression we sought for.

Lemma 7.8. *For γ an admissible curve and $z \in \mathbb{C}$ with $\operatorname{Re} z < 0$ the complex path integral*
$$\frac{1}{2\pi \mathrm{i}} \int_\gamma \lambda^z R(\lambda, A) \, \mathrm{d}\lambda \in \mathscr{L}(X)$$
converges in operator norm locally uniformly in $\{z : \operatorname{Re} z < 0\}$, and is independent of γ.

Proof. The integrand is holomorphic, and since
$$\|\lambda^z R(\lambda, A)\| \leq \frac{M|\lambda|^{\operatorname{Re} z} \mathrm{e}^{\pi |\operatorname{Im} z|}}{1 + |\lambda|} \tag{7.2}$$
holds, it follows that the integral is absolutely and locally uniformly convergent.

The independence of the integral from γ follows from Cauchy's integral theorem and from the estimate above. $\square$

The next result shows that our new definition for the power would be consistent with the usual one.

Proposition 7.9. *For $n \in \mathbb{N}$ and $z = -n$ we have*
$$A^z = A^{-n} = \frac{1}{2\pi \mathrm{i}} \int_\gamma \lambda^{-n} R(\lambda, A) \, \mathrm{d}\lambda.$$

Proof. We may assume that γ is an admissible curve of the form given in Example 7.7. Let us consider the part of γ inside of $\mathrm{B}(0,r)$ that we close on the left by a circle arc around 0 of radius r. Hence we obtain the closed curve γ_r. The residue theorem applied to

$$\lambda^{-n}R(\lambda,A)=\sum_{k=0}^{\infty}(-1)^k\lambda^{k-n}(-A)^{-k+1}=-\sum_{k=0}^{\infty}\lambda^{k-n}A^{-k+1}$$

yields

$$\frac{1}{2\pi\mathrm{i}}\int\limits_{\gamma_r}\lambda^{-n}R(\lambda,A)\,\mathrm{d}\lambda=A^{-n},$$

since γ_r is negatively oriented. If we let $r\to\infty$ we obtain the assertion by the estimate in (7.2). $\square$

Now we can create a definition out of what we have seen.

Definition 7.10. For $z\in\mathbb{C}$ with $\operatorname{Re}z<0$ define the operator

$$A^z:=\frac{1}{2\pi\mathrm{i}}\int\limits_{\gamma}\lambda^zR(\lambda,A)\,\mathrm{d}\lambda. \tag{7.3}$$

We call A^z the **power** of A.

As one might have expected we have the following algebraic property.

Proposition 7.11. *For $z,w\in\mathbb{C}$ with $\operatorname{Re}z,\operatorname{Re}w<0$ we have*

$$A^zA^w=A^{z+w}.$$

Proof. Take two admissible curves γ and $\tilde\gamma$ such that γ lies to the left of $\tilde\gamma$. Then we have

$$A^z=\frac{1}{2\pi\mathrm{i}}\int\limits_{\gamma}\lambda^zR(\lambda,A)\,\mathrm{d}\lambda\quad\text{and}\quad A^w=\frac{1}{2\pi\mathrm{i}}\int\limits_{\tilde\gamma}\mu^wR(\mu,A)\,\mathrm{d}\mu.$$

We calculate the product of the two powers

$$A^zA^w=\frac{1}{(2\pi\mathrm{i})^2}\int\limits_{\tilde\gamma}\int\limits_{\gamma}\lambda^z\mu^wR(\mu,A)R(\lambda,A)\,\mathrm{d}\lambda\,\mathrm{d}\mu=\frac{1}{(2\pi\mathrm{i})^2}\int\limits_{\tilde\gamma}\int\limits_{\gamma}\frac{\lambda^z\mu^w}{\lambda-\mu}\big(R(\mu,A)-R(\lambda,A)\big)\,\mathrm{d}\lambda\,\mathrm{d}\mu$$

by the resolvent identity. We can continue by Fubini's theorem

$$\begin{aligned}&=\frac{1}{2\pi\mathrm{i}}\int\limits_{\tilde\gamma}R(\mu,A)\frac{1}{2\pi\mathrm{i}}\int\limits_{\gamma}\frac{\lambda^z\mu^w}{\lambda-\mu}\,\mathrm{d}\lambda\,\mathrm{d}\mu-\frac{1}{2\pi\mathrm{i}}\int\limits_{\gamma}R(\lambda,A)\frac{1}{2\pi\mathrm{i}}\int\limits_{\tilde\gamma}\frac{\lambda^z\mu^w}{\lambda-\mu}\,\mathrm{d}\mu\,\mathrm{d}\lambda\\&=\frac{1}{2\pi\mathrm{i}}\int\limits_{\tilde\gamma}\mu^z\mu^wR(\mu,A)\,\mathrm{d}\mu-0=A^{w+z},\end{aligned}$$

where we also used Cauchy's integral theorem. $\square$

Before turning to the full definition including positive fractional powers we study the properties of A^z as a function of $z\in\{w:\operatorname{Re}w<0\}$.

Proposition 7.12. *The mapping*

$$\{w : \operatorname{Re} w < 0\} \ni z \mapsto A^z \in \mathscr{L}(X)$$

is holomorphic.

Proof. Since the integrand in (7.3) is holomorphic and since the integral is locally uniformly convergent in $\{w : \operatorname{Re} w < 0\}$ (see Lemma 7.8), the assertion follows immediately. $\square$

The next result provides important formulas in which the path integral is replaced by integration on the real line. To ensure convergence at 0 we need to have a condition on the exponent.

Proposition 7.13. *For $z \in \mathbb{C}$ with $-1 < \operatorname{Re} z < 0$ we have*

$$A^z = \frac{\sin(\pi z)}{\pi} \int_0^\infty s^z R(-s, A)\,\mathrm{d}s = -\frac{\sin(\pi z)}{\pi} \int_0^\infty s^z (s + A)^{-1}\,\mathrm{d}s. \tag{7.4}$$

Proof. Choose the admissible curve γ as in Example 7.7. Then

$$\begin{aligned}
A^z &= \frac{1}{2\pi\mathrm{i}} \int_\gamma \lambda^z R(\lambda, A)\,\mathrm{d}\lambda \\
&= -\frac{1}{2\pi\mathrm{i}} \int_0^\infty (s\mathrm{e}^{\mathrm{i}\theta} + a)^z R(s\mathrm{e}^{\mathrm{i}\theta} + a, A)\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}s + \frac{1}{2\pi\mathrm{i}} \int_0^\infty (s\mathrm{e}^{-\mathrm{i}\theta} + a)^z R(s\mathrm{e}^{-\mathrm{i}\theta} + a, A)\mathrm{e}^{-\mathrm{i}\theta}\,\mathrm{d}s \\
&= -\frac{1}{2\pi\mathrm{i}} \int_0^\infty \mathrm{e}^{\mathrm{i}\theta(z+1)}(s + a\mathrm{e}^{-\mathrm{i}\theta})^z R(s\mathrm{e}^{\mathrm{i}\theta} + a, A)\,\mathrm{d}s + \frac{1}{2\pi\mathrm{i}} \int_0^\infty \mathrm{e}^{-\mathrm{i}\theta(z+1)}(s + \mathrm{e}^{\mathrm{i}\theta}a)^z R(s\mathrm{e}^{-\mathrm{i}\theta} + a, A)\,\mathrm{d}s.
\end{aligned}$$

If we let $a \to 0$ and $\theta \nearrow \pi$, then we obtain

$$= -\frac{1}{2\pi\mathrm{i}} \int_0^\infty \mathrm{e}^{\mathrm{i}\pi(z+1)} s^z R(-s, A)\,\mathrm{d}s + \frac{1}{2\pi\mathrm{i}} \int_0^\infty \mathrm{e}^{-\mathrm{i}\pi(z+1)} s^z R(-s, A)\,\mathrm{d}s = \frac{\sin(\pi z)}{\pi} \int_0^\infty s^z R(-s, A)\,\mathrm{d}s.$$

This passage to the limit is allowed, since we can estimate the integrand

$$\|\mathrm{e}^{\mathrm{i}\theta(z+1)}(s + a\mathrm{e}^{-\mathrm{i}\theta})^z R(s\mathrm{e}^{\mathrm{i}\theta} + a, A)\| \le K \frac{s^{\operatorname{Re} z}}{1 + s},$$

which is integrable near $s = 0$ since $\operatorname{Re} z > -1$ and is integrable near ∞ since $\operatorname{Re} z < 0$. Hence we can apply Lebesgue's dominated convergence theorem. $\square$

A trivial consequence of this theorem is the identity

$$a^\alpha = -\frac{\sin(\pi a)}{\pi} \int_0^\infty \frac{s^\alpha}{s + a}\,\mathrm{d}s \tag{7.5}$$

for $a \in (-1, 0)$. This is one of the scalar identities motivating the formulas behind fractional powers of operators.

Proposition 7.14. *For $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha \in (-1,0)$ we have*

$$\|A^\alpha\| \le M\frac{|\sin(\pi\alpha)|}{\sin(\pi\operatorname{Re}\alpha)}.$$

In particular, the mapping

$$(-1,0) \ni \alpha \mapsto A^\alpha \in \mathscr{L}(X)$$

is uniformly bounded.

Proof. We use the representation (7.4) from Proposition 7.13. For $\alpha \in (-1,0)$ we have

$$\|A^\alpha\| = \Big\|\frac{\sin(\pi\alpha)}{\pi}\int_0^\infty s^\alpha R(-s,A)\,\mathrm{d}s\Big\| \le \frac{|\sin(\pi\alpha)|}{\pi}\int_0^\infty s^{\operatorname{Re}\alpha}\frac{M}{1+s}\,\mathrm{d}s = M\frac{|\sin(\pi\alpha)|}{\sin(\pi\operatorname{Re}\alpha)},$$

by identity (7.5). For real α the assertion follows from this trivially. □

Corollary 7.15. *If A is densely defined, then $T(t) := A^{-t}$, $t > 0$ and $T(0) = I$ defines a strongly continuous semigroup.*

Proof. The mapping T has the semigroup property by Proposition 7.11. Since $T : [0,1] \to \mathscr{L}(X)$ is bounded by Proposition 7.14, it suffices to check the strong continuity at 0 on the dense subspace $D(A)$ (see Proposition 2.5). For $t \in (0,1)$ and $f \in D(A)$ we have by Proposition 7.13 and by identity (7.5) for $a = 1$ that

$$A^{-t}f - f = \frac{\sin(\pi t)}{\pi}\int_0^\infty s^{-t}\big(R(-s,A) - R(-s,I)\big)\,\mathrm{d}s = \frac{\sin(\pi t)}{\pi}\int_0^\infty \frac{s^{-t}}{1+s}R(-s,A)(I-A)f\,\mathrm{d}s.$$

From this it follows

$$\|A^{-t}f - f\| \le M\frac{\sin(\pi t)}{\pi}\int_0^\infty \frac{s^{-t}}{1+s}\|R(-s,A)\|\cdot\|(I-A)f\|\,\mathrm{d}s \le \frac{\sin(\pi t)}{\pi}\int_0^\infty \frac{s^{-t}}{(1+s)^2}\,\mathrm{d}s\|(I-A)f\|,$$

which converges to 0 as $t \searrow 0$. □

7.2 Complex powers

We expect that $A^z = (A^{-z})^{-1}$ should hold, so A^z should be injective. The first result tells that this intuition—unlike many others concerning complex powers—is true.

Proposition 7.16. *For $z \in \mathbb{C}$ with $\operatorname{Re} z < 0$ the operator A^z is injective.*

Proof. Let $n \in \mathbb{N}$ be such that $-n < \operatorname{Re} z$ and take $w := -n - z$. Then we have

$$A^zA^w = A^wA^z = A^{z+w} = A^{-n}.$$

By Proposition 7.9, the operator A^{-n} is the n^{th} power of the inverse A^{-1} of A so it is injective, hence so are A^z and A^w. □

The result above allows us to formulate the next definition.

Definition 7.17. Let $z \in \mathbb{C}$. If $\operatorname{Re} z < 0$, then the operator A^z is defined in (7.3). If $\operatorname{Re} z > 0$, then we set

$$D(A^z) := \operatorname{ran}(A^{-z}) \quad \text{and} \quad A^z := (A^{-z})^{-1},$$

which exists by Proposition 7.16. If $\operatorname{Re} z = 0$, then we define

$$D(A^z) := \big\{ f \in X : A^{z-1} f \in D(A) \big\} \quad \text{and} \quad A^z f := A A^{z-1} f.$$

In particular, we set $A^0 = I$. The operator A^z is called the **complex power** of A.

First, we study algebraic properties of the complex powers A^z.

Proposition 7.18. *a) For $z \in \mathbb{C}$ with $\operatorname{Re} z < -n$, $n \in \mathbb{N}$ we have that*

$$\operatorname{ran}(A^z) \subseteq D(A^n) \quad \textit{and} \quad A^n A^z f = A^{n+z} f \quad \textit{for all } f \in X.$$

b) For $z \in \mathbb{C}$ with $\operatorname{Re} z < 0$, $f \in D(A^n)$, $n \in \mathbb{N}$ we have

$$A^z f \in D(A^n) \quad \textit{and} \quad A^z A^n f = A^n A^z f.$$

c) For $z \in \mathbb{C}$ with $0 \le \operatorname{Re} z < n$ we have

$$D(A^z) = \big\{ f \in X : A^{z-n} f \in D(A^n) \big\} \quad \textit{and} \quad A^z f = A^n A^{z-n} f.$$

Proof. a) We first prove the assertion for $n = 1$ and assume $\operatorname{Re}(z) < -1$. Let γ be an admissible curve. Since

$$\|\lambda^z A R(\lambda, A)\| = \|\lambda^z (\lambda R(\lambda, A) - I)\| \le (M_0 + 1) |\lambda|^{\operatorname{Re} z} \mathrm{e}^{\pi |\operatorname{Im} z|},$$

and since $\lambda^z A R(\lambda, A)$ is bounded on compact parts of γ, we see that the integral A^z converges in the norm of $\mathscr{L}(X, D(A))$. Since A is closed we obtain

$$A A^z = \frac{1}{2\pi \mathrm{i}} \int_\gamma \lambda^{z+1} R(\lambda, A) \, \mathrm{d}\lambda - \frac{1}{2\pi \mathrm{i}} \int_\gamma \lambda^z \, \mathrm{d}\lambda.$$

By closing γ on the right by large circle arc of radius $r > 0$ and by letting $r \to \infty$, we see that integral on the right hand side is 0 by Cauchy's integral theorem. Hence the assertion follows.

For general $n \in \mathbb{N}$ we can argue inductively. Indeed, let $n \in \mathbb{N}$, $n \ge 2$, and let $z \in \mathbb{C}$ be with $\operatorname{Re} z < -n$. Then $\operatorname{Re}(z + 1) < -(n - 1)$, hence $\operatorname{ran}(A^{z+1}) \subseteq D(A^{n-1})$ and $A^{n-1} A^{z+1} f = A^{n+z} f$ follows for $f \in X$ by the induction hypothesis. We already proved $A^{z+1} = A A^z$. From these the assertion follows.

b) Since $R(\lambda, A)$ and A^n commute on $D(A^n)$, it follows that A^z and A^n commute on $D(A^n)$. This implies the assertion.

c) If $\operatorname{Re} z > 0$, then $D(A^z) = \operatorname{ran}(A^{-z})$. By Proposition 7.11 we have $A^{-n} = A^{-z} A^{z-n}$. Hence $f \in \operatorname{ran}(A^{-z})$ if and only if $A^{z-n} f \in \operatorname{ran}(A^{-n}) = D(A^n)$ and the asserted equality follows. Suppose $\operatorname{Re} z = 0$, and notice that the assertion is true for $n = 1$ by the definition of A^z. For $n \in \mathbb{N}$, $n \ge 2$ we have

$$A^{z-n} = A^{1-n} A^{z-1}.$$

This implies that $A^{z-n} f \in D(A^n)$ if and only if $A^{z-1} \in D(A)$. $\square$

The next result is the extension of the "semigroup property" from Proposition 7.11.

Proposition 7.19. *For $z, w \in \mathbb{C}$ with $\operatorname{Re} z < \operatorname{Re} w$ the following assertions are true:*

a) One has $D(A^w) \subseteq D(A^z)$ and $A^z f = A^{z-w} A^w f$ for all $f \in D(A^w)$

b) For every $f \in D(A^w)$ we have $A^z f \in D(A^{w-z})$ and $A^w f = A^{w-z} A^z f$.

c) If $f \in D(A^z)$ and $A^z f \in D(A^{w-z})$, then $f \in D(A^w)$.

Proof. a) Let $n \in \mathbb{N}$ satisfy $n > \operatorname{Re} w$, and let $f \in D(A^w)$, then by Proposition 7.18.c) we have $A^{w-n} f \in D(A^n)$. Proposition 7.11 yields $A^{-n+z} f = A^{z-w} A^{-n+w} f \in D(A^n)$, so actually again by Proposition 7.18.c) we conclude $x \in D(A^z)$.

b) Let $f \in D(A^w)$ and let $n \in \mathbb{N}$ satisfy $n > \operatorname{Re} w$ and $n > \operatorname{Re} w - \operatorname{Re} z$. Then we can write

$$A^{-n+w-z} A^w f = A^{-n+w-z} A^{z-w} A^w f = A^{-n} A^w f,$$

hence by Proposition 7.18.c) we obtain $A^z \in D(A^{w-z})$ and $A^{w-z} A^z f = A^w f$.

c) Take $f \in D(A^z)$ such that $A^z f \in D(A^{w-z})$. Let $n \in \mathbb{N}$ satisfy $n > \operatorname{Re} w$ and $n > \operatorname{Re} w - \operatorname{Re} z$. Proposition 7.11 yields

$$A^{w-2n} f = A^{w-n-z} A^{z-n} f = A^{w-n-z} A^{-n} A^z f = A^{-n} A^{w-z-n} A^z f.$$

By Proposition 7.18.c) the right-hand side belongs to $D(A^{2n})$, so again this proposition gives $f \in D(A^w)$. The equality

$$A^w f = A^{2n} A^{w-2n} f = A^{2n} A^{-n} A^{w-z-n} A^z f = A^{w-z} A^z f$$

also follows. □

7.3 Domain embeddings

As mentioned in the introduction, our main interest in powers of operators lies in the excellent properties of their domains. Hence, we turn to study various norms on $D(A^z)$ for $\operatorname{Re} z > 0$.

Proposition 7.20. *a) For $z \in \mathbb{C}$ the operator A^z is closed.*

b) For $\operatorname{Re} z > 0$ the graph norm of A^z is equivalent to

$$\|f\|_{A^z} := \|A^z f\| \quad \textit{for all } f \in D(A^z).$$

c) For $z, w \in \mathbb{C}$ with $0 \le \operatorname{Re} z < \operatorname{Re} w$ the embedding

$$D(A^w) \hookrightarrow D(A^z)$$

is continuous.

Proof. a) If $\operatorname{Re}(z) \neq 0$, either A^z or A^{-z} is bounded, hence both of them are closed. If $\operatorname{Re}(z) = 0$, then $A^z = AA^{z-1}$, where A^{z-1} is bounded. By Exercise 1 the product is closed.

b) Since A^z has bounded inverse A^{-z}, we have $\|f\| \le \|A^{-z}\| \cdot \|A^z f\|$. From this it follows that the graph norm is equivalent to $\|\cdot\|_{A^z}$.

c) By Proposition 7.19.a) we have $D(A^w) \subseteq D(A^z)$ and

$$A^{z-w} A^w f = A^z f \quad \text{for all } f \in D(A^w),$$

hence $\|A^z\| \le \|A^{z-w}\| \cdot \|A^w\|$. □

To be able to relate the various norms $\|\cdot\|_{A^\alpha}$ more precisely, we need the next alternative formula for complex powers.

Proposition 7.21 (Balakrishnan's formula). *For $\alpha \in \mathbb{C}$ with $0 < \operatorname{Re}\alpha < 1$ we have*

$$A^\alpha f = \frac{\sin(\pi\alpha)}{\pi} A \int_0^\infty s^{\alpha-1}(s+A)^{-1} f \,\mathrm{d}s \quad = \frac{\sin(\pi\alpha)}{\pi} A \int_0^\infty s^{\alpha-1}(s+A)^{-1} f \,\mathrm{d}s \quad \textit{for all } f \in D(A).$$

Proof. Since $-1 < \operatorname{Re}\alpha - 1 < 0$ we obtain from (7.4) in Proposition 7.13

$$A^{\alpha-1} f = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty s^{\alpha-1}(s+A)^{-1} f \,\mathrm{d}s. \tag{7.6}$$

Since $s^{\alpha-1}(s+A)^{-1} f \in D(A)$ for every $s > 0$ and since

$$\int_0^\infty s^{\alpha-1}(s+A)^{-1} A f \,\mathrm{d}s$$

is a convergent improper integral, the closedness of A implies that the right-hand side in (7.6) belongs to $D(A)$ and that

$$AA^{\alpha-1} f = \frac{\sin(\pi\alpha)}{\pi} A \int_0^\infty s^{\alpha-1}(s+A)^{-1} f \,\mathrm{d}s = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty s^{\alpha-1}(s+A)^{-1} A f \,\mathrm{d}s.$$

By Proposition 7.19.a) we have $A^\alpha f = AA^{\alpha-1} f$, hence the statement is proved. □

Remark 7.22. The above proof can be modified to yield the following more general statement: For $\alpha, \beta \in \mathbb{C}$ with $0 < \operatorname{Re}\alpha < \operatorname{Re}\beta \leq 1$ we have

$$A^\alpha f = \tfrac{\sin(\pi(\beta-\alpha))}{\pi} \int_0^\infty s^{\alpha-\beta}(s+A)^{-1} A^\beta f \,\mathrm{d}s = \tfrac{\sin(\pi(\beta-\alpha))}{\pi} \int_0^\infty s^{\alpha-\beta}(s+A)^{-1} A^\beta f \tag{7.7}$$

for all $f \in D(A^\beta)$

We can make use of this representation to obtain finer relations between the $\|\cdot\|_{A^\alpha}$ norms.

Proposition 7.23. *For $\alpha, \beta \in \mathbb{C}$ with $0 < \operatorname{Re}\alpha < \operatorname{Re}\beta < 1$ there is $K_0 \geq 0$ such that the following assertions holds:*

a) For all $f \in D(A^\beta)$

$$\|A^\alpha f\| \leq K_0\big(t^{\operatorname{Re}\alpha - \operatorname{Re}\beta + 1}\|f\| + t^{\operatorname{Re}\alpha - \operatorname{Re}\beta}\|A^\beta f\|\big) \quad \textit{for all } t > 0. \tag{7.8}$$

b) For all $f \in D(A^\beta)$

$$\|A^\alpha f\| \leq 2K_0 \|f\|^{\operatorname{Re}\beta - \operatorname{Re}\alpha} \cdot \|A^\beta f\|^{1-(\operatorname{Re}\beta - \operatorname{Re}\alpha)}. \tag{7.9}$$

Proof. For $f \in D(A^\beta)$ we have by Remark 7.22 that

$$
\begin{aligned}
\|A^\alpha f\| &\le \tfrac{|\sin(\pi(\beta-\alpha))|}{\pi}\Big(\int_0^t \|s^{\alpha-\beta}A^\beta(s+A)^{-1}\|\cdot\|f\|\,\mathrm{d}s + \int_t^\infty \|s^{\alpha-\beta}(s+A)^{-1}\|\cdot\|A^\beta f\|\,\mathrm{d}s\Big)\\
&\le \tfrac{|\sin(\pi(\beta-\alpha))|}{\pi}\Big(\int_0^t \|s^{\alpha-\beta}A^{\beta-1}A(s+A)^{-1}\|\cdot\|f\|\,\mathrm{d}s + \int_t^\infty \|s^{\alpha-\beta}(s+A)^{-1}\|\cdot\|A^\beta f\|\,\mathrm{d}s\Big)\\
&\le \tfrac{|\sin(\pi(\beta-\alpha))|}{\pi}\Big(\|A^{\beta-1}\|\int_0^t s^{\mathrm{Re}\,\alpha-\mathrm{Re}\,\beta}\Big(1+\frac{Ms}{1+s}\Big)\,\mathrm{d}s\|f\| + \int_t^\infty s^{\mathrm{Re}\,\alpha-\mathrm{Re}\,\beta}\frac{M}{s+1}\,\mathrm{d}s\|A^\beta f\|\Big)\\
&\le \tfrac{|\sin(\pi(\beta-\alpha))|}{\pi}\Big(t^{\mathrm{Re}\,\alpha-\mathrm{Re}\,\beta+1}(1+M)\|A^{\beta-1}\|\cdot\|f\| + Mt^{\mathrm{Re}\,\alpha-\mathrm{Re}\,\beta}\|A^\beta f\|\Big)\\
&\le K_0\big(t^{\mathrm{Re}\,\alpha-\mathrm{Re}\,\beta+1}\|f\| + t^{\mathrm{Re}\,\alpha-\mathrm{Re}\,\beta}\|A^\beta f\|\big).
\end{aligned}
$$

This proves assertion a).

For $f = 0$ the desired inequality (7.9) is trivial. For $f \neq 0$ set $t = \frac{\|A^\beta f\|}{\|f\|}$ in the inequality above to conclude

$$\|A^\alpha f\| \le 2K_0\|f\|^{\mathrm{Re}\,\beta-\mathrm{Re}\,\alpha}\|A^\beta f\|^{1-(\mathrm{Re}\,\beta-\mathrm{Re}\,\alpha)}. \qquad \square$$

Remark 7.24. The proof above works whenever $A^{\beta-1}$ is bounded, for example also for $\beta = 1$. In particular, we obtain for $\alpha \in [0,1]$

$$\|A^\alpha f\| \le K\|f\|^{1-\alpha}\|Af\|^\alpha \quad \text{for all } f \in D(A), \tag{7.10}$$

the limiting cases $\alpha = 0$ and $\alpha = 1$ being trivial.

With some more work one can prove the following general version of interpolation type inequalities, which we mention here without proof.

Theorem 7.25 (Moment inequality)**.** *For $\alpha < \beta < \gamma$ there is $K \ge 0$ such that*

$$\|A^\beta f\| \le K\|A^\alpha f\|^{\frac{\gamma-\beta}{\gamma-\alpha}}\cdot\|A^\gamma f\|^{\frac{\beta-\alpha}{\gamma-\alpha}} \quad \textit{holds for all } f \in D(A^\gamma).$$

Corollary 7.26. *Let $\alpha \in (0,1]$ and let B be a closed operator such that $D(B) \supseteq D(A^\alpha)$. Then the following assertions are true:*

a) There is a $K \ge 0$ such that

$$\|Bf\| \le K\|A^\alpha f\| \quad \textit{for all } f \in D(A^\alpha).$$

b) There is $K_0 \ge 0$ such that

$$\|Bf\| \le K_0\big(s^\alpha\|f\| + s^{\alpha-1}\|Af\|\big) \quad \textit{holds for all } s > 0 \textit{ and } f \in D(A^\alpha).$$

Proof. a) Since by Exercise 1 the operator $BA^{-\alpha}$ is closed, and since it is by assumption everywhere defined, it is bounded by the closed graph theorem, see Theorem 2.32. Boundedness of $BA^{-\alpha}$ means precisely the assertion.

b) The assertion follows from part a) and Proposition 7.23.a). $\square$

After having seen the fine structure of the embeddings of domains of complex powers, let us close this lecture by returning to the motivating question. To which the last result gives one possible answer.

Proposition 7.27. *Let A generate a semigroup T of type (M,ω) with $\omega < 0$ and consider the powers $(-A)^z$ for $\operatorname{Re} z > 0$. The domain $D((-A)^z)$ is invariant under the semigroup T. The restriction of T to this subspace is a strongly continuous semigroup of bounded linear operators for the norm $\|\cdot\|_{(-A)^z}$. The type of this semigroup is (M,ω).*

Proof. Since the bounded operator $(-A)^{-z}$ commutes with $-R(-\lambda,-A) = R(\lambda, A)$, as a consequence of the convergence of the implicit Euler scheme(see Theorem 5.10), we obtain that $(-A)^{-z}$ commutes with the semigroup operators $T(t)$. This implies that $\operatorname{ran}((-A)^{-z}) = D((-A)^z)$ is invariant under the semigroup. Moreover, we have

$$\|(-A)^z T(t) f\| \leq \|T(t)\| \cdot \|(-A)^z f\|,$$

so $T(t) \in \mathscr{L}(D((-A)^z))$. The strong continuity follows from

$$\|(T(t)-I)f\|_{(-A)^z} = \|(-A)^z (T(t)-I) f\| = \|T(t)(-A)^z f - (-A)^z f\|. \qquad \square$$

Exercises

1. Suppose $A : D(A) \to X$ is closed and $B \in \mathscr{L}(X)$ is bounded.

a) Prove that the product AB with

$$D(AB) = \big\{f \in X : Bx \in D(A)\big\}.$$

is closed.

b) Give an example for A and B such that BA with $D(BA) = D(A)$ is not closed.

2. Let $m = (m_n) \subseteq \mathbb{C}$ be a sequence. Give a sufficient and necessary condition on m so that the multiplication operator M_m fulfills Assumption 7.3. Determine in that case the powers of M_m.

3. Prove that for $t \in \mathbb{R}$ and $f \in D(A^{\mathrm{i}t})$ we have $A^{\mathrm{i}t} f \in D(A^{-\mathrm{i}t})$ and $A^{-\mathrm{i}t} A^{\mathrm{i}t} f = f$.

4. Prove the identity (7.7) in Remark 7.22.

5. Suppose A is densely defined, and take $z \in \mathbb{C}$ with $\operatorname{Re} z < 0$. Prove that $T(t) := A^{zt}$, $t > 0$ and $T(0) = I$ defines a strongly continuous semigroup.

6. Assume we have proved assertion b) in Proposition 7.23. Deduce part a) from that.

7. Let $\alpha \in (0,1)$. Prove that for all $\lambda > 0$ sufficiently large we have

$$\|A^\alpha R(-\lambda, A)\| < 1.$$

Compare this to Exercise 6.5.

8. Prove what has been remaining from Proposition 7.1.