Solutions of Exercises 11

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Exercise 1

Prove the reccurence relation

$$\phi_j(hC)f = \frac{1}{j!}f + hC\phi_{j+1}(hC)f \tag{1}$$

for all j=0,1,2, and $f \in X$.

Proof. Fix $h > 0$. For $j = 0$, we have by definition

$$\phi_1(hC)f = hC\frac{1}{h}\int_0^h e^{(h-\tau)C}f\ d\tau = C\int_0^h e^{sC}f\ ds = e^{hC}f - f,$$

where we used Lemma 2.9 in the last equality. Since $\phi_0(hC) := e^{hC}$, (1) for $j = 0$ follows.
Now, let $j > 0$ and $f \in D(C)$. Since $(s \mapsto e^{sA}f)$ is continuously differentiable with derivative Ce^{sA}, using integration by parts, we obtain

$$\begin{aligned}\phi_j(hC)f &= \frac{1}{h^j}\int_0^h \frac{\tau^{j-1}}{(j-1)!}e^{(h-\tau)C}f\ d\tau \\ &= \frac{1}{j!}f + \frac{1}{h^j}\int_0^h \frac{\tau^j}{j!}Ce^{(h-\tau)C}f\ d\tau \\ &= \frac{1}{j!}f + C\frac{1}{h^j}\int_0^h \frac{\tau^j}{j!}e^{(h-\tau)C}f\ d\tau,\end{aligned}$$

where the last equality holds since C is closed. Therefore, (1) holds for $f \in D(C)$. For arbitrary $f \in X$, let $(f_n) \subset D(C)$ be a sequence converging to f. From above, we know

$$\phi_j(hC)f_n = \frac{1}{j!}f_n + hC\phi_{j+1}(hC)f_n \qquad \forall n \in \mathbb{N}.$$

This implies, because $\phi_j(hC)$ is bounded, that $hC\phi_{j+1}(hC)f_n$ converges to $\phi_j(hC)f - \frac{1}{j!}f$ as $n \to \infty$. Therefore, by boundedness of $\phi_{j+1}(hC)$, we have

$$\phi_{j+1}(hC)f_n \to \phi_{j+1}(hC)f, \qquad hC\phi_{j+1}(hC)f_n \to \phi_j(hC)f - \frac{1}{j!}f,$$

as $n \to \infty$. Since C is closed, we conclude that $\phi_{j+1}(hC)f \in D(C)$ and

$$hC\phi_{j+1}(hC)f = \phi_j(hC)f - \frac{1}{j!}f.$$

□

Exercise 2

Let C be the operator from Example 11.5. Prove that the H^4-norm makes $D(C^2)$ a Banach space.

Proof. It suffices to show that $D(C^2)$ is a closed subspace of $(H^4(\Omega), \|.\|_{H^4})$. Let $f_n \in D(C^2)$ and $f_n \overset{H^4}{\to} f$ (this denotes convergence in H^4) as $n \to \infty$. Clearly, this implies that $f_n \overset{H^2}{\to} f$ and $f_n \overset{H^1}{\to} f$. Since $f_n \in D(C) = H_0^1(\Omega) \cap H^2(\Omega)$, it follows that $f \in D(C)$ because $H_0^1(\Omega)$ and $H^2(\Omega)$ are Banach spaces.
Now, we show $Cf_n \overset{H^2}{\to} Cf$. Since $a, b \in C^3(\bar{\Omega})$, one easily sees that for $f \in D(C)$

$$Cf = (A + B)f = \partial_x(a\partial_x f) + \partial_y(b\partial_y f) \in H^2.$$

In fact, for instance

$$\|\partial_{xx}\partial_x(a\partial_x f)\|_{L^2(\Omega)} \leq \sum_{i=0}^{3} \binom{3}{i} \left\| (\frac{\partial^i}{\partial x^i}a) \cdot \frac{\partial^{4-i}}{\partial x^{4-i}}f \right\|_{L^2(\Omega)} \leq M\|f\|_{H^4} < \infty$$

Here,

$$M = C \max_{i=1,..3} \left\| (\frac{\partial^i}{\partial x^i}a)^2 \right\|_\infty \tag{2}$$

which exists since $a \in C^3(\bar{\Omega})$ and where $C > 0$ is a suitable constant. Analogously, we deduce

$$\|Cf_n - Cf\|_{H^2(\Omega)} \leq \sum_{(\alpha,\beta)\in\mathbb{N}\times\mathbb{N}, \alpha+\beta\leq 4} M_{\alpha,\beta} \|\frac{\partial^\alpha \partial^\beta}{\partial x^\alpha \partial y^\beta}(f - f_n)\|_{L^2(\Omega)} \leq \tilde{M}\|f - f_n\|_{H^4},$$

where $M_{\alpha,\beta}$ is computed similarly as in (2). Since $f_n \overset{H^4}{\to} f$, it follows that $Cf_n \overset{H^2}{\to} Cf$ and $Cf_n \overset{H^1}{\to} Cf$. Thus, $Cf \in H_0^1(\Omega) \cap H^2(\Omega) = D(C)$. Therefore, $f \in D(C^2)$.

□

Exercise 3

Consider the operators A, B and C from Example 11.5 with $a = b = 1$. Show that they generate an analytic contraction semigroups.

Proof. Since for $a = b = 1$ the operators A and B are only 'working' on either x or y, this operators can be considered on $L^2(0,1)$ instead of $L^2(\Omega)$. From lecture 1 and 2 we know that they generate the heat semigroup on $L^2(0,1)$, which is a contraction semigroup. To show that we even get an analytic semigroup, we apply Proposition 9.19. Therefore, it remains to show that there exist an $\alpha \in (0, \pi/2)$ such that $e^{-i\alpha}A$ and $e^{i\alpha}A$ generate bounded semigroups. Since for $f \in D(A)$, we obtain using the integration by parts formula that

$$\begin{aligned}\langle e^{\pm i\alpha} Af, f\rangle_{L^2(0,1)} &= e^{\pm i\alpha} \int 0^1 f''(t)\overline{f(t)}\, dt \\ &= e^{\pm i\alpha} f'\overline{f}\big|_0^1 - e^{\pm i\alpha} \int_0^1 f'(t)\overline{f'(t)}\, dt \\ &= -\, e^{\pm i}\alpha \|f'\|^2_{L^2(0,1)}.\end{aligned}$$

For $\alpha \in (0, \pi/2)$, we deduce that $\mathrm{Re}\langle e^{\pm i\alpha} Af, f\rangle_{L^2(0,1)} \leq 0$, hence $e^{\pm i\alpha}A$ is dissipative. Clearly, $e^{\pm i\alpha}A$ are densely defined, closed and their resolvent sets include $(0, \infty)$, because these properties hold for A. Thus, by the Lumer-Phillips theorem (lecture 6), these operators generate a contraction semigroup and, hence, by Prop. 9.19, we conclude that A generates a (bounded) analytic semigroup. Obviously, the argumentation for B is the same.

For C, we have seen a similar result in lecture 2 (the heat semigroup on $L^2(\mathbb{R})$ is a contraction semigroup). In fact, to show that C ($X = L^2(\Omega)$) is the generator of a contraction semigroup, we use the Lumer-Phillips theorem once again. Apparentely, the domain of C is dense in $L^2(\Omega)$ and the operator is closed (since C is the closure of its restriction to $C^2(\Omega)$). Furthermore, from PDEswe know (see e.g.[1]) that for $g \in C^2(\Omega)$ the elliptic equation

$$-\Delta f + f = g, \qquad f = 0 \text{ on } \partial\Omega,$$

has a solution in $C^2(\overline{\Omega})$. Hence, $\mathrm{rg}(\mathrm{I} - C)$ is dense. Finally, for $f \in D(C)$ we make a similar observation as above,

$$\begin{aligned}\langle e^{\pm i\alpha} Cf, f\rangle_{L^2(\Omega)} &= e^{\pm i\alpha} \int_{\partial\Omega} f(\nabla f \cdot \nu)\, d\nu - e^{\pm i\alpha} \int_{\Omega} \nabla f \cdot \overline{\nabla f}\, dx \\ &= -\, e^{\pm i\alpha} \|\nabla f\|^2_{L^2(\Omega)},\end{aligned}$$

which has real-part less or equal zero for $\alpha \in (0, \pi/2)$ (and also for $\alpha = 0$). By Lumer-Phillips, we conclude that $e^{\pm i\alpha}C$ (and A) generate contraction semigroups, thus, C is the generator of a bounded(contraction) analytic semigroup by Prop. 9.19. □

[1] Evans, R., *Partial Differential Equations.* Graduate Studies in Mathematics, American Mathematical Society, 1998.

Exercise 4

Suppose A generates a contraction semigroup on the Hilbert space H. Prove that the Cayley transform

$$G = (\mathrm{I} + A)(\mathrm{I} - A)^{-1}$$

is a contraction, i.e. $\|G\| \leq 1$.

Proof. We begin with the following fact, which is easy to see

$$G = (\mathrm{I} + A)(\mathrm{I} - A)^{-1} = -\frac{\mathrm{I} - A}{\mathrm{I} - A} + \frac{2\,\mathrm{I}}{\mathrm{I} - A} = -\mathrm{I} + 2(\mathrm{I} - A)^{-1}.$$

Using this, we see, for $x \in X$, that

$$\|Gx\|^2 = \|x\|^2 + 4\|(\mathrm{I} - A)^{-1}x\| - 4\,\mathrm{Re}\langle(\mathrm{I} - A)^{-1}x, x\rangle. \tag{3}$$

Since A generates a contraction semigroup, A is dissipative (by Hille-Yoshida Theorem). Let us note, as seen in lecture 6, that dissipativity is equivalent to

$$\mathrm{Re}\langle y, Ay\rangle \leq 0 \qquad \forall y \in D(A).$$

This implies

$$\|y\|^2 - \mathrm{Re}\langle y, (\mathrm{I} - A)y\rangle \leq 0. \tag{4}$$

By defining $y = (\mathrm{I} - A)^{-1}x$, (4) can be used to estimate the right hand side in (3), which reads then

$$\|Gx\|^2 \leq \|x\|^2.$$

Thus, $\|G\| \leq 1$. □

Exercise 5

Prove Theorem 11.14

Theorem. *The Marchuk-Strang splitting is convergent at time level $t > 0$ if the stability condition (11.10) holds for the approximate semigroups, and the approximate generators satisfy Assumption 11.7.*

Proof. Basically, the proof is very similar to one done for the sequential splitting. Recall that the Marchuk-Strang splitting is defined by

$$F_m = T_m(h/2)S_m(h)S_m(h/2),$$

where T_m and S_m denote the approximate semigroups. As in the lecture, let A_m, B_m be the corresponding generators respectively and T, S (A, B) be the 'exact' semigroups (generator). Furthermore, let the closure C of $A + B$ generate a semigroup. As for the sequential splitting, in order to show the convergence, we are going to apply the Modified Chernoff Theorem, Theorem 11.11. Clearly, $F_m(h)$ is a bounded operator and $F_m(0) = \mathrm{I}$ for all $m \in \mathbb{N}$, $h \geq 0$. Furthermore, the following stability

$$\|(F_m(h))^k\| \leq Me^{h\omega k}$$

holds by Assumption 11.7. and Exercise 5 of lecture 10. Therefore, to apply Theorem 11.11, it remains find $\lambda > 0$ and a dense subspace Y such that for $f \in Y$

$$\lim_{m\to\infty} \frac{J_m F_m P_m f - J_m P_m f}{h}$$

exists uniformly in $h \in (0, t_0]$ (for some $t_0 > 0$) and that

$$Gf := \lim_{h\to 0^+} \lim_{m\to\infty} \frac{J_m F_m P_m f - J_m P_m f}{h} \tag{5}$$

exists. Furthermore, $(\lambda\,\mathrm{I} - G)Y$ should be dense.
For that, we consider some fixed $t_0 > 0$ and

$$\begin{aligned}
\frac{1}{h}\big[J_m F_m(h) P_m f - J_m P_m f\big] &= \frac{J_m T_m(h/2) S_m(h) T_m(h/2) P_m f - J_m P_m f}{h} \\
&= J_m T_m(h/2) S_m(h) P_m \frac{1}{2} \frac{J_m T_m(h/2) P_m f - J_m P_m f}{h/2} + \\
&\quad + J_m T_m(h/2) P_m \frac{J_m S_m(h) P_m f}{h} - \frac{1}{h} J_m P_m f \\
&= \underbrace{J_m T_m(h/2) S_m(h) P_m \frac{1}{2} \frac{J_m T_m(h/2) P_m f - J_m P_m f}{h/2}}_{=\alpha(m,h)} + \\
&\quad + \underbrace{J_m T_m(h/2) P_m \frac{J_m S_m(h) P_m f - J_m P_m f}{h}}_{=\beta(m,h)} + \\
&\quad + \underbrace{\frac{1}{2} \frac{J_m T_m(h/2) P_m f - J_m P_m f}{h/2}}_{=\gamma(m,h)}
\end{aligned}$$

where we used that $P_m J_m = \mathrm{I}$ several times. Now, the argument is similar as in the proof for the sequential splitting: From Lemma 11.12 we know that for $f \in D(A)$,

$$\lim_{m\to\infty} \frac{J_m T_m(h/2) P_m f - J_m P_m f}{h/2} = \frac{1}{(h/2)}[T(h/2)f - f], \tag{6}$$

as $m \to \infty$ uniformly in $h \in (0, t_0]$. Analogously, for $f \in D(B)$,

$$\lim_{m\to\infty} \frac{J_m S_m(h) P_m f - J_m P_m f}{h} = \frac{1}{h}[S(h)f - f], \tag{7}$$

as $m \to \infty$ uniformly in $h \in (0, t_0]$. Let $f \in D(A) \cap D(B)$. To see the convergence of $\beta(m,h)$ we consider

$$\begin{aligned}\|\beta(m,h) - Bf\| &= \|J_m T_m(h/2) P_m \frac{1}{h}\big[J_m S_m(h) P_m - J_m P_m\big] - J_m T_m(h/2) P_m Bf + \\ &\quad + J_m T_m(h/2) P_m Bf - Bf\| \\ &\leq \|J_m T_m(h/2) P_m\| \left\| \frac{J_m S_m(h) P_m - J_m P_m}{h} - Bf \right\| + \\ &\quad + \|J_m T_m(h/2) P_m Bf - Bf\|.\end{aligned}$$

Since $\|J_m T_m(h/2) P_m\| \leq C e^{(t_0/2)\omega}$ for all $m \in \mathbb{N}$ and $h \in [0, t_0]$ and because of (7), the first term converges to zero as $m \to \infty$ and $h \to 0^+$ (the limit for $m \to \infty$ is uniformly in h). By Remark 11.8 and the strong continuity, this convergence also holds for the second term.
Similarly, $\alpha(m,h) \to (1/2)Af$ can be seen. For $\gamma(m,h)$ we can apply (6) directly.
Altogether, we obtain that

$$\lim_{m\to\infty} \alpha(h,m), \quad \lim_{m\to\infty} \beta(h,m) \quad \text{and} \quad \lim_{m\to\infty} \gamma(m,h)$$

exist uniformly in $h \in (0, t_0]$ and that

$$\lim_{h\to 0^+} \lim_{m\to\infty} (\alpha(h,m) + \beta(h,m) + \gamma(m,h)) = \frac{1}{2}Af + Bf + \frac{1}{2}Af.$$

Therefore, for $f \in D(A) \cap D(B)$

$$\lim_{m\to\infty} \frac{1}{h}\big[J_m F_m(h) P_m f - J_m P_m f\big]$$

exists uniformly in $h \in (0, t_0]$ and

$$\lim_{h\to 0^+} \lim_{m\to\infty} \frac{1}{h}\big[J_m F_m(h) P_m f - J_m P_m f\big] = Af + Bf = Cf.$$

Hence, we set $Y = D(A) \cap D(B)$ which is dense by assumption. Since C generates a semigroup, there exists a positive λ such that $(\lambda \mathrm{I} - C)$ is boundedly invertible. Thus, $(\lambda \mathrm{I} - C)Y$ is also dense (in fact, assume that there exists a non-empty open set $O \subseteq X$ such that $O \cap (\lambda \mathrm{I} - C)Y = \emptyset$. Therefore, and since $(\lambda \mathrm{I} - C)$ is boundedly invertible, $(\lambda \mathrm{I} - C)^{-1} O$ is a non-empty open set with $(\lambda \mathrm{I} - C)^{-1} O \cap Y = \emptyset$. This contradicts that Y is dense.)

□