DISCONTINUOUS TRANSITION IN 2D POTTS: I. ORDER-DISORDER INTERFACE CONVERGENCE

MORITZ DOBER, ALEXANDER GLAZMAN, AND SÉBASTIEN OTT

Abstract. It is known that the planar q-state Potts model undergoes a discontinuous phase transition when $q > 4$ and there are exactly $q+1$ extremal Gibbs measures at the transition point: q ordered (monochromatic) measures and one disordered (free). We focus on the Potts model under the Dobrushin order–disorder boundary conditions on a finite $N \times N$ part of the square grid. It was previously known that, if defined at all, the interface between the ordered and disordered sides of the box exhibits unbounded fluctuations. Our main result is that this interface is a well defined object, has $\sqrt{N}$ fluctuations, and converges to a Brownian bridge under diffusive scaling. The same holds also for the corresponding FK-percolation model for all $q > 4$.

Our proofs rely on a coupling between FK-percolation, the six-vertex model, and the random cluster representation of an Ashkin–Teller model (ATRC), and on a detailed study of the latter. Fine mixing properties the ATRC model are derived using the link to the six vertex model and its height function, while the coupling transfers the study of the interface in FK-percolation to the study of long subcritical clusters in the ATRC model. These are then studied via the development of a "renewal picture" *à la* Ornstein-Zernike. Along the way, we derive various properties of the Ashkin-Teller model, such as Ornstein-Zernike asymptotics for its two-point function.

In a companion work, we provide a detailed study of the Potts model under order-order Dobrushin conditions. In particular, we show that the model is subject to the phenomenon of wetting, and derive the scaling limit of the interface (under diffusive scaling).

[AG: This is preliminary version, soon to be replaced by an arXiv version.] ★

Contents

Date: December 21, 2024.

1. Introduction and results

The Potts model is a classical model of statistical mechanics introduced in 1952 [Pot52]. Each vertex of a graph is assigned one of q states (colours), with states of adjacent vertices interacting with a strength depending on the temperature $T > 0$ of the system. At $q = 2$, this corresponds to the seminal Ising model. The Potts model becomes increasingly ordered as the temperature decreases, and a phase transition occurs on lattices $\mathbb{Z}^d$ with $d \geq 2$ at some transition temperature $T_c(q, d) > 0$. Depending on q and d, the transition is either discontinuous (first-order) or continuous (higher-order). Such a rich behaviour has brought a lot of attention to the Potts model.

Interface in the planar Potts model. Our work is restricted to dimension $d = 2$. In this case, planar duality and a correlation inequality (available when $q \geq 1$) have allowed to a watershed of progress in the phase diagram of the Potts model in the last two decades:

- the transition occurs at the self-dual point $T_c(q) := [\log(1 + \sqrt{q})]^{-1}$ [BD12];
- the transition is continuous when $q = 2, 3, 4$, in a sense that, at $T_c(q)$, there is a unique Gibbs measure [DST17] (see also [GL23] for another argument);
- the transition is discontinuous when $q > 4$ [DGH+21] (see also [RS20] for a short proof), and any Gibbs measure can be written as a linear combination of $q + 1$ extremal Gibbs measures (q monochromatic and one free) [GM23].

We focus on discontinuous transitions ($q > 4$) and study interfaces at $T_c(q)$ separating different states. The structure of extremal Gibbs measures (described above) leads to two natural definitions of Dobrushin boundary conditions:

- *order-disorder*: one half of the boundary is of a fixed colour and the other one is free (no colour assigned);
- *order-order*: both halves of the boundary are assigned different fixed colours.

The phenomenology is quite different in the two cases.

The main result of the current paper is convergence of the (properly defined) interface under order-disorder conditions to the Brownian bridge in the diffusive limit. Results of such precision were previously proven for $T < T_c(q)$ and the order-order interface. Indeed, planar duality transfers this problem to the study of the typical geometry of long clusters at $T > T_c$, for which a quasi-renewal structure was developed in [CIV08]. We do use similar ideas, but the situation is more involved: duality does not help directly as we work at the self dual point, and we consider the order-disorder interface. Instead, we use a sequence of combinatorial mappings relating FK-percolation to the Ashkin-Teller model, and use planar duality there to map the problem to the study of suitable long subcritical clusters, for which we derive results similar to those of [CIV08]. We then transport back the convergence result to the FK and Potts models. The coupling between the different models and the study of the AT model builds on [GP23, ADG24], while the renewal picture for long subcritical clusters builds on [CIV03, CIV08, AOV24].

Let us also mention the work [MMSRS91] that showed $\sqrt{N}$ fluctuations when q is taken to be large enough. Our results imply this for all $q > 4$. In this generality, the recent work [GM23] implies that the interface (if defined at all) exhibits diverging fluctuations.

The case of the order-order conditions is addressed in our companion paper: we show emergence of a free layer in the middle of width $\sqrt{N}$ (*wetting*) and establish convergence of its boundaries to two Brownian bridges conditioned not to intersect.

In particular, we prove that in both cases of Dobrushin conditions when $q > 4$ the scaling limit of the Potts model (i.e.: as the mesh of the lattice goes to 0) is a straight line separating two "constant" regions. To compare with what happens in the case of continuous phase transition, let us describe the expected behaviour for $q = 2, 3, 4$. The mesmerizing physics conjecture from 1980s postulates conformal invariance. Schramm's [Sch00] geometric interpretation of this conjecture asserts that as one takes the scaling limit of the Potts model, the interface converges to a random fractal curve called the Schramm–Loewner Evolution. This is has been proven rigorously only at $q = 2$ (Ising model) by Smirnov et al [Smi10, CS11].

The "Ornstein-Zernike" (OZ) theory is a (non-rigorous) picture introduced in [OZ14, Zer16] of how do correlations functions in various models behave. Their main idea was to postulate a suitable *renewal structure* satisfied by correlation functions, which leads to very precise expressions for the asymptotics of the said correlations. The first rigorous implementation of this renewal picture was done by Abraham and Kunz [AK77] using perturbative expansions. The modern approach, which started with the work of Chayes, and Chayes [CC86] on self avoiding walk and of Campanino, Chayes, and Chayes [CCC91] on Bernoulli percolation, is based on creating a renewal structure *à la* OZ for elongated subcritical objects (SAW or percolation clusters containing a distant point). Both these works rely on heavy combinatorial study, and the renewal steps are "irreducible crossings of slabs".

This idea was further developed by Ioffe [Iof98] for the ballistic phase of the self-avoiding walk, introducing key measure-tilting ideas coming from large deviation theory. His strategy was then extended to Bernoulli percolation in [CI02], to the high temperature of the Ising model in [CIV03], and to FK-percolation [CIV08]. Compared to the cases of SAW and Bernoulli percolation (where measure factorizes nicely), the structure obtained in the case of Ising and FK, while still geometrically being a concatenation of "irreducible blocs", is not a real renewal structure, only a sequence of "fast mixing kernels", which study is heavier and performed in [CIV03]. The last step to finally obtain a true renewal structure from the fast mixing kernel picture was done in [OV18].

These works on subcritical clusters of FK-percolation all take place in any dimensions. But, in the 2 dimensional case (for $\mathbb{Z}^2$), planar duality allows to re-write the study of the interface of Potts model at $T < T_c(q)$ as the asymptotic study of a percolation cluster at $T > T_c(q)$ conditioned to contain $(0, 0)$ and $(N, 0)$.

The contribution of the present work to this line of works is to derive a "renewal picture" for the long clusters of the Random Cluster representation of the Ashkin–Teller model (in dimension 2), in an intermediate regime. This model is then related to the Potts interface using a suitable adaptation of a coupling introduced in [GP23].

We now formally state our main results for the Potts model, the FK percolation and the Ashkin–Teller model.

1.1. **Potts model.** For $i, j \in \mathbb{Z}^2$, we write $i \sim j$ if i and j are adjacent, i.e. $|i-j| = 1$, and denote by $\mathbb{E}$ the set of pairs of adjacent points. We view $\mathbb{Z}^2$ both as a set of points on the plane having integer coordinates and as a graph (square grid) with edges linking points at distance one. Denote by $\mathbb{E}$ the set of edges in $\mathbb{Z}^2$ and write $i \sim j$ if $\{i, j\} \in \mathbb{E}$.

Let $G = (V, E)$ be a subgraph of $\mathbb{Z}^2$. Take parameters $T > 0$, $q \in \{2, 3, 4, \dots\}$, and boundary conditions $\eta \in \{0, 1, \dots, q\}^{\mathbb{V}}$. The Potts model on G with boundary conditions η is the probability measure on $\{1, \dots, q\}^V$ given by

$$\mathsf{Potts}^{\eta}_{G;T,q}(\sigma) := \tfrac{1}{Z_{\mathsf{Potts}}} \cdot \exp\left[\tfrac{1}{T} \cdot \left(\sum_{\{i,j\}\in E} \delta(\sigma_i, \sigma_j) + \sum_{i\in V, j\in V^c:\, i\sim j} \delta(\sigma_i, \eta_j)\right)\right],$$

where $Z^{\eta}_{\mathsf{Potts}} = Z^{\eta}_{\mathsf{Potts}}(V, T, q, \eta)$ is the unique normalising constant (called *partition function*) that renders the above a probability measure. Note that we allowed the boundary conditions to take the value 0 (which is not an allowed value for the spins) to mimic boundary conditions favoring none of the q possible states of the spins.

We say that η defines the order-disorder (1-free) Dobrushin boundary conditions (and denote it by 1/f) if $\eta((x, y)) = \mathbb{1}_{\geq 0}(y)$. Identify $\Lambda_n := \{-n, \dots, n\}^2$ with the induced subgraph of $\mathbb{Z}^2$ on this set of vertices. The Dobrushin boundary conditions on Λ_n impose existence of an interface between color one and the rest. This can be made explicit, but for brevity we choose to define directly the upper and lower discrete envelopes of this interface: $\Gamma^{+}_{\mathsf{Potts}}$ and $\Gamma^{-}_{\mathsf{Potts}}$ respectively. Given $\sigma \in \{1, \dots, q\}^{\Lambda_n}$, define $\bar\sigma \in \{0, 1, \dots, q\}^{\mathbb{Z}^2}$ to be its extension outside of Λ_n by the Dobrushin boundary conditions: one on $\mathbb{Z} \times \mathbb{Z}_{\geq 0}$ and zero on $\mathbb{Z} \times \mathbb{Z}_{<0}$. For $k = -n, \dots, n$, define

$$\Gamma^{+,n}_{\mathsf{Potts}}(k) := \max\{y \in \mathbb{Z} :\ (k, y-1) \xleftrightarrow{\bar\sigma\neq 1} (\mathbb{Z} \times \mathbb{Z}_{<0}) \setminus \Lambda_n\},$$
$$\Gamma^{-,n}_{\mathsf{Potts}}(k) := \min\{y \in \mathbb{Z} :\ (k, y+1) \xleftrightarrow{\bar\sigma= 1} (\mathbb{Z} \times \mathbb{Z}_{\geq 0}) \setminus \Lambda_n\},$$

where $(k, y-1) \xleftrightarrow{\bar\sigma\neq 1} (\mathbb{Z} \times \mathbb{Z}_{<0}) \setminus \Lambda_n$ states the existence of a path in $\mathbb{Z}^2$ in going from $(k, y-1)$ to $(\mathbb{Z}\times\mathbb{Z}_{<0})\setminus\Lambda_n$ and consisting of vertices where $\bar\sigma \neq 1$ and $(k, y) \xleftrightarrow{\bar\sigma=1,\text{ diag}} (\mathbb{Z} \times \mathbb{Z}_{\geq 0}) \setminus \Lambda_n$ states the existence of a path in $\mathbb{Z}^2$ *with diagonal connectivity* going from (k, y) to $(\mathbb{Z}\times\mathbb{Z}_{\geq 0})\setminus\Lambda_n$ and consisting of vertices where $\bar\sigma = 1$. By $\mathbb{Z}^2$ with diagonal connectivity we mean a graph with the vertex-set $\mathbb{Z}^2$ with edges linking vertices at distance at most $\sqrt{2}$. Define the rescaled linear interpolation of $\Gamma^{+,n}_{\mathsf{Potts}}$ and $\Gamma^{-,n}_{\mathsf{Potts}}$ by

$$\tilde\Gamma^{\pm,n}_{\mathsf{Potts}}(t) := \tfrac{1}{\sqrt{n}}\big((1 - \{2tn-n\})\Gamma^{\pm}_{\mathsf{Potts}}(\lfloor 2tn-n\rfloor) + \{2tn-n\}\Gamma^{\pm}_{\mathsf{Potts}}(\lceil 2tn-n\rceil)\big),$$

where $\lfloor\,\rfloor$, $\lceil\,\rceil$, $\{\,\}$ denote respectively lower rounding, upper rounding, and fractional part.

Theorem 1. *Let $q > 4$ be integer and take $T = T_c(q)$. For $n \in \mathbb{N}$, sample $\Gamma^{\pm,n}_{\mathsf{Potts}}$ and $\tilde\Gamma^{\pm,n}_{\mathsf{Potts}}$ from $\mathsf{Potts}^{1/\mathrm{f}}_{\Lambda_n;T_c(q),q}$ as described above. Then, as n tends to infinity,*

(1) both $\big(\tilde\Gamma^{+,n}_{\mathsf{Potts}}(t)\big)_{t\in[0,1]}$ and $\big(\tilde\Gamma^{-,n}_{\mathsf{Potts}}(t)\big)_{t\in[0,1]}$ converge in law to $(c_q \mathrm{b}_t)_{t\in[0,1]}$, where b_t is a standard Brownian bridge and $c_q > 0$ is some constant;

(2) the probability that $\max_k \big|\Gamma^{+,n}_{\mathsf{Potts}}(k) - \Gamma^{-,n}_{\mathsf{Potts}}(k)\big| \geq 2\ln(n)^9$ tends to zero.

1.2. **FK percolation.** The main tool in analyzing the Potts model is the Fortuin–Kasteleyn (FK) percolation (or random-cluster) model [FK72] that allows to express the spin-spin correlation function via connection probabilities. We first define it and then state the relation between the two models. Take a finite subgraph $G = (V, E)$ of $\mathbb{Z}^2$, parameters $p \in (0, 1)$, $q > 0$ and boundary conditions $\xi \in \{0, 1\}^{\mathbb{E}}$. We identify any $\omega \in \{0, 1\}^{\mathbb{E}}$ with the set of edges $e \in \mathbb{E}$ for which $\omega_e = 1$ (*open edges*) and with the spanning subgraph of $\mathbb{Z}^2$ defined by the open edges. The FK-percolation model on G with boundary conditions ξ is the probability measure on $\{0, 1\}^{\mathbb{E}}$ given by

$$\mathsf{FK}^{\xi}_{G;p,q}(\eta) := \tfrac{1}{Z_{\mathsf{FK}}} \cdot p^{|\eta\cap E|}(1-p)^{|E\setminus\eta|}\, q^{\kappa_V(\eta)}\, \mathbb{1}_{\eta=\xi \text{ on } \mathbb{E}\setminus E},$$

where $Z_{\mathsf{FK}} = Z_{\mathsf{FK}}(G, p, q, \xi)$ is the partition function and $\kappa_V(\eta)$ is the number of connected component (*clusters*) of $(\mathbb{Z}^2, \eta)$ that intersect V.

The *free* and *wired* measures correspond to the choices $\xi \equiv 0$ and $\xi \equiv 1$, respectively, and we simply write f and w instead of ξ, respectively. We will be interested in the Dobrushin wired/free boundary conditions: $\xi_e = 1$ if and only if $e \subset \mathbb{Z} \times \mathbb{Z}_{\geq 0}$ (see Figure 1). We denote these boundary conditions by 1/0.

For $n \geq 1$, define the graph $G_n = (V_n, E_n)$ by

$$E_n := \{e \in \mathbb{E} : e \subset \Lambda_n\} \cup \{e \in \mathbb{E} : e \subset \mathbb{Z} \times \mathbb{Z}_{\geq 0} \text{ and } e \cap \Lambda_n \neq \varnothing\}, \quad V_n := \bigcup_{e \in E_n} e.$$

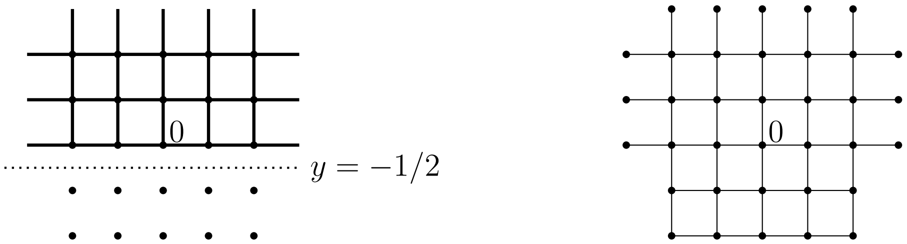

FIGURE 1. Left: wired-free Dobrushin boundary conditions. Right: the graph G_2.

The seminal Edwards–Sokal coupling [ES88] states that, when $q \geq 2$ is integer and $p = 1 - \exp\left[-\frac{1}{T} \cdot \frac{q}{q-1}\right]$, coloring clusters of $\omega \sim \mathsf{FK}^{1/0}_{G_n;p,q}$ independently in colors $1, 2, \ldots, q$ gives a spin configuration $\sigma \sim \mathsf{Potts}^{1/\mathrm{f}}_{\Lambda_n;T,q}$. We will denote this coupling between the Potts model and the FK percolation by $\mathsf{ES}^{1/0}_{\Lambda_n,T,q}$.

We define an interface in the FK percolation forced by the Dobrushin boundary conditions using planar duality. Note that the lattice dual to $\mathbb{Z}^2$ is again a square lattice and we denote it by $(\mathbb{Z}^2)^*$. For each edge e of $\mathbb{Z}^2$, denote by e^* the edge of $(\mathbb{Z}^2)^*$ that is dual to e, i.e. the unique edge of $(\mathbb{Z}^2)^*$ that intersects e; see Fig. 2. Given $\omega \in \{0, 1\}^{\mathbb{E}}$, define its dual by

$$\omega^*_{e^*} := 1 - \omega_e.$$

The FK percolation at $p_c(q)$, given by

$$p_c(q) := 1 - \exp\left[-\frac{1}{T_c(q)} \cdot \frac{q}{q-1}\right] = \frac{\sqrt{q}}{\sqrt{q}+1},$$

is known to enjoy the self-duality: if $\omega \sim \mathsf{FK}^{1/0}_{G_n;T_c(q),q}$, then its dual ω^* is also distributed as an FK-percolation with parameters q and $p_c(q)$, but on a dual graph and under dual Dobrushin boundary conditions. This self-dual nature is also revealed when looking at the loop representation of the FK percolation model. Specifically, we draw two arcs next to every primal or dual edge of ω, as shown on Fig. 3. These arcs link together into loops separating primal and dual clusters and one interface tracing the boundary of the union of primal clusters attached to the upper boundary of Λ_n. Denote this interface by Γ_{FK}. Remarkably, a standard application of the Euler's formula, allows to rewrite the distribution of ω via loops which are symmetric with respect primal and dual configurations:

$$\mathsf{FK}^{1/0}_{G_n;p_c(q),q}(\omega) = \frac{1}{Z_{\mathsf{loop}}} \cdot \sqrt{q}^{\#\,\text{loops}}, \tag{1}$$

where # loops stands for the number of loops in the loop representation of ω. This is a part of the classical Baxter–Kelland–Wu [BKW76] coupling between the FK percolation and the six-vertex model; see Sections 4 and 5.1.4.

As for the Potts model, we define the upper and the lower discrete envelops of Γ_{FK}:

$$\Gamma^{+,n}_{\mathsf{FK}}(k) := \max\{y \in \mathbb{Z} : \ (k \pm \tfrac{1}{2}, y - \tfrac{1}{2}) \overset{\omega^*}{\longleftrightarrow} (\{\mathbb{Z} + \tfrac{1}{2}\} \times \{\mathbb{Z}_{<0} + \tfrac{1}{2}\}) \setminus \Lambda^*_n\},$$
$$\Gamma^{-,n}_{\mathsf{FK}}(k) := \min\{y \in \mathbb{Z} : \ (k, y+1) \overset{\omega}{\longleftrightarrow} (\mathbb{Z} \times \mathbb{Z}_{\geq 0}) \setminus \Lambda_n\},$$

where $\Lambda^*_n := [-n,n]^2 \cap (\mathbb{Z}^2)^*$. The rescaled linear interpolations $\tilde{\Gamma}^{\pm,n}_{\mathsf{FK}}$ of $\Gamma^{\pm,n}_{\mathsf{FK}}$ are defined in the same way as in the Potts model. Our main result for the FK percolation is an invariance principle for $\tilde{\Gamma}^{\pm,n}_{\mathsf{FK}}$:

Theorem 2. *Let $q > 4$ be a real number and take $p = p_c(q)$. For $n \in \mathbb{N}$, sample $\Gamma^{\pm,n}_{\mathsf{FK}}$ and $\tilde{\Gamma}^{\pm,n}_{\mathsf{FK}}$ from $\mathsf{FK}^{1/0}_{G_n;p_c(q),q}$ as described above. Then, as n tends to infinity, the convergence results from Theorem 1 hold for $\Gamma^{\pm,n}_{\mathsf{FK}}$ and $\tilde{\Gamma}^{\pm,n}_{\mathsf{FK}}$ in place of $\Gamma^{\pm,n}_{\mathsf{Potts}}$ and $\tilde{\Gamma}^{\pm,n}_{\mathsf{Potts}}$.*

We draw attention to the precision of our control of the interface. The interface Γ^n_{FK} at $p_c(q)$ exhibits linear fluctuations when $q \in [1,4]$ (continuous transition); it was expected that this is not the case when $q > 4$ (discontinuous transition), but remained open. Our results imply that this is indeed the case and, moreover, the probability that Γ^n_{FK} exhibits linear fluctuations is in fact exponentially small.

Our proof goes via developing the Ornstein–Zernike for the Ashkin–Teller model. We proceed by introducing the latter model and stating our results for it.

1.3. **Ashkin–Teller model.** Introduced in 1943 [AT43] as a generalization of the Ising model to a four-component system, the Ashkin–Teller (AT) model can be viewed as a pair of interacting Ising models. Take a finite subgraph $G = (V,E)$ of $\mathbb{Z}^2$, parameters $J, U > 0$ and boundary conditions $\sigma, \sigma' \in \{0, \pm 1\}^{\mathbb{Z}^2}$. The AT model on G with parameters $J, U \in \mathbb{R}$ and boundary conditions (σ, σ') is a probability measures on pairs $\tau, \tau' \in \{\pm 1\}^V$ given by

$$\mathsf{AT}^{\sigma,\sigma'}_{G,J,U}(\tau,\tau') = \tfrac{1}{Z_{\mathsf{AT}}} \cdot \exp\Bigg[\sum_{\{i,j\}\in E} J(\tau_i\tau_j + \tau'_i\tau'_j) + U\tau_i\tau_j\tau'_i\tau'_j$$
$$+ \sum_{i\in V, j\in V^c:\ i\sim j} J(\tau_i\sigma_j + \tau'_i\sigma'_j) + U\tau_i\sigma_j\tau'_i\sigma'_j\Bigg]$$

where $Z_{\mathsf{AT}} = Z_{\mathsf{AT}}(G,J,U,\sigma,\sigma')$ is the partition function. Taking $\sigma = \sigma' \equiv 1$ we obtain plus-plus boundary conditions and taking $\sigma = \sigma' \equiv 0$ we obtain free-free boundary conditions; we denote the corresponding measures by $\mathsf{AT}^{+,+}_{G,J,U}$ and $\mathsf{AT}^{\mathrm{f,f}}_{G,J,U}$ respectively.

We consider only $J > 0$, since flipping the sign of J corresponds to flipping the sign of τ' and τ at one of the two partite classes of $\mathbb{Z}^2$. Using the Ising-duality for τ' and then for τ, the *self-dual curve* of the parameters was identified [MS71]:

$$\sinh 2J = e^{-2U}. \tag{2}$$

The correlations are monotone [KS68] along the lines of a constant ratio J/U. The case $U = 0$ gives two independent Ising models and the line $J = U$ is in direct correspondence with the four-state Potts model and the FK percolation with the cluster-weight $q = 4$. In addition, the three models are related on the self-dual line (2), with $q > 4$ corresponding to $U > J$. A precise coupling was constructed in [GP23] via the six-vertex (square ice) model based on the works of Fan [Fan72a] and Wegner [Weg72] and on the seminal Baxter–Kelland–Wu (BKW) correspondence [BKW76];

see also [HDJS13]. This coupling is crucial to the current work and is described in detail in Sections 4.3 and 5.1.4.

Fix $U > J > 0$ that satisfy $\sinh 2J = e^{-2U}$ and omit them from the notation. Correlation inequalities [KS68] guarantee the existence of the infinite-volume limits $\mathsf{AT}^{\mathrm{f,f}}$ and $\mathsf{AT}^{+,+}$ of the free and monochromatic AT measures $\mathsf{AT}_G^{\mathrm{f,f}}$ and $\mathsf{AT}_G^{+,+}$, respectively. Our first result states that their marginals on the single spin τ coincide.

Proposition 1.1. *In the above notation, the measures $\mathsf{AT}^{\mathrm{f,f}}$ and $\mathsf{AT}^{+,+}$ have the same marginal distribution on τ.*

Denote the expectation operators with respect to $\mathsf{AT}^{\mathrm{f,f}}$ and $\mathsf{AT}^{+,+}$ by $\langle \cdot \rangle^{\mathrm{f,f}}$ and $\langle \cdot \rangle^{+,+}$, respectively, and define the inverse correlation length ν by setting, for $x \in \mathbb{R}^2$,

$$\nu(x) := -\lim_{n\to\infty} \tfrac{1}{n} \ln\langle \tau_0 \tau_{\lfloor nx \rfloor}\rangle^{\mathrm{f,f}} = -\lim_{n\to\infty} \tfrac{1}{n} \ln\langle \tau_0 \tau_{\lfloor nx \rfloor}\rangle^{+,+},$$

where $\lfloor \cdot \rfloor$ is the componentwise integer part. The existence of the limit is derived in a standard manner from correlation inequalities and a subadditive argument. In [ADG24], it was shown that $\mathsf{AT}_{\Lambda_n}^{+,+}$ admits exponential decay of correlations in τ, which implies that $\nu > 0$. The following theorem establishes sharp Ornstein–Zernike-type asymptotics for the 2-point function.

Theorem 3. *The inverse correlation length ν is a norm on $\mathbb{R}^2$. Furthermore, uniformly in $|x| \to \infty$,*

$$\langle \tau_0 \tau_x \rangle^{\mathrm{f,f}} = \langle \tau_0 \tau_x \rangle^{+,+} = \tfrac{\Psi(x/|x|)}{\sqrt{|x|}}\, e^{-\nu(x)}\, (1 + o(1)),$$

where Ψ is a strictly positive analytic function on $\mathbb{S}^1$.

1.4. **Summary of the paper: what is new?** The main novelty of our work is the use of the graphical (or random-cluster) representation of the Ashkin–Teller model (ATRC model) to study the FK percolation and the Potts models. At a first glance, the FK percolation model has been by now much better understood, see the classical book [Gri06] and more recent lecture notes [DC17]. Moreover, the FK model has been used to establish some basic properties for the ATRC model [GP23, ADG24]. However, compared to the FK percolation at its transition point when $q > 4$, the ATRC model at its self-dual line when $U > J$ remarkably exhibits a unique Gibbs measure. This brings more symmetries that play a key role for developing the OZ theory.

This comes at some cost: the ATRC model is supported on *pairs* of edge configurations. Thus, the domain Markov property is significantly weaker than in the FK percolation defined on a single edge configuration. In particular, it is highly non-trivial to prove mixing properties of the ATRC model. For this we use a coupling of the ATRC model to the six-vertex model, whose height function enjoys additional monotonicity properties. Using the classical work of Alexander [Ale98] and a new general mixing result [Ott25], we prove the exponential ratio mixing for the ATRC model.

We point out a technical issue that the FK percolation model under standard Dobrushin boundary conditions is directly coupled only to the ATRC with rather involved boundary conditions (we call it a modified ATRC model). Fortunately, apart from the boundary, this model still satisfies the FKG lattice condition. This implies mixing estimates and allows us to develop the renewal theory.

Organisation of the article. We now provide the structure of the paper.

Section 2: notation and conventions that will be used throughout the article.

Section 3: definition of the ATRC model, its basic properties and the results that we establish for this model, including the uniqueness of the ATRC Gibbs measure and strong mixing properties.

Section 4: coupling between the FK percolation and modified ATRC models. The coupling is very sensitive to boundary conditions, eg. note appearance of a different boundary-cluster weight in [GP23]. We extend this coupling to standard Dobrushin boundary conditions in the FK percolation at a price of rather inconvenient conditions in the ATRC model.

Section 5: proof of new mixing properties for the ATRC model and, in particular, uniqueness of the ATRC Gibbs measure. We adapt the classical works of Alexander [Ale92, Ale98, Ale04] and use the recent works on the ATRC model [GP23, ADG24].

Section 6: derivation of a strong mixing from a weak mixing using a general argument from [Ott25].

Section 7: development of the OZ theory in the infinite volume following [CIV08] (see also [Ott19]) and using the mixing established in Section 5. Specifically, we show that the interface in the ATRC model can be well-approximated by a certain directed random walk.

Section 8: adaptation of the approximation by a random walk from Section 7 to the finite-volume setting and the proof the invariance principle for the modified ATRC model following [GI05].

Section 9: the invariance principle for FK percolation and the Potts models is derived from that for the modified ATRC model, utilising the coupling from Section 4.

Acknowledgements We want to say big thanks to Ioan Manolescu for frequent discussions that started already in 2019-20 and for pointing out that it is enough to work with tall rectangles. A part of the work was accomplished during the visits of some of us to the Universities of Fribourg, Geneva, Innsbruck and Vienna and at the NCCR SwissMAP research station in Les Diablerets. We want to thank these institutions and all people involved for their hospitality and creating excellent working conditions. This research was funded in whole or in part by the Austrian Science Fund (FWF) 10.55776/P34713.

2. Notations and conventions

2.1. **Graphs and lattices.** Let $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ be a graph. We simply write $xy = \{x, y\}$ for an edge $\{x, y\} \in \mathbb{E}$. Given finite subsets $\Lambda \subset \mathbb{V}$ and $E \subset \mathbb{E}$, define

$$\mathbb{E}_\Lambda := \{e \in \mathbb{E} : e \subset \Lambda\}, \quad \mathbb{V}_E := \bigcup_{e \in E} e,$$

$$\partial^{\text{in}}\Lambda := \{x \in \Lambda : \exists y \in \mathbb{V} \setminus \Lambda,\, xy \in \mathbb{E}\}, \quad \partial^{\text{ex}}\Lambda := \{y \in \mathbb{V} \setminus \Lambda : \exists x \in \Lambda,\, xy \in \mathbb{E}\},$$

$$\partial^{\text{in}}E := \{e \in E : \exists f \in \mathbb{E} \setminus E, e \cap f \neq \varnothing\}, \quad \partial^{\text{ex}}E := \{e \in \mathbb{E} \setminus E : \exists f \in E, e \cap f \neq \varnothing\},$$

$$\partial^{\text{edge}}\Lambda := \{xy \in \mathbb{E} : x \in \Lambda, y \in \mathbb{V} \setminus \Lambda\}, \quad \overline{\Lambda} := \Lambda \cup \partial^{\text{ex}}\Lambda, \quad \overline{E} := E \cup \partial^{\text{ex}}E.$$

In case of ambiguity, we add $\mathbb{G}$ as a subscript (and write, for example, $\partial^{\text{in}}_{\mathbb{G}}\Lambda$) to emphasise that the boundary is taken in $\mathbb{G}$. The *interior* of Λ (in $\mathbb{G}$) is given by $\Lambda \setminus \partial^{\text{in}}\Lambda$. The sub-graph *induced* by $\Lambda \subset \mathbb{V}$ is given by $(\Lambda, \mathbb{E}_\Lambda)$. We say that Λ is *simply connected* if both the sub-graphs induced by Λ and by $\mathbb{V} \setminus \Lambda$ are connected.

We will mainly work on $\mathbb{Z}^2$ with nearest-neighbour edges, and on its dual. We will denote the *primal lattice* by $\mathbb{L}_\bullet = \mathbb{Z}^2$ and its dual by $\mathbb{L}_\circ = (1/2, 1/2) + \mathbb{Z}^2$. Denote by $\mathbb{E}^\bullet$ the nearest-neighbour edges between sites in $\mathbb{L}_\bullet$ (the primal edges), by $\mathbb{E}^\circ$ the nearest-neighbour edges between sites in $\mathbb{L}_\circ$ (the dual edges).

Define the upper and lower half planes by

$$\mathbb{H}^+ := \mathbb{R} \times \mathbb{R}_{\geq 0}, \quad \mathbb{H}^- := \mathbb{R} \times \mathbb{R}_{<0},$$

and, for $n, m \geq 0$, set

$$\Lambda_{n,m} := \{-n, \dots, n\} \times \{-m, \dots, m\}, \quad \partial^{\pm}_{n,m} := \partial^{\mathrm{ex}} \Lambda_{n,m} \cap \mathbb{H}^{\pm}.$$

Moreover, define the graph $G_{n,m} = (V_{n,m}, E_{n,m})$ by

$$V_{n,m} = \Lambda_{n,m} \cup \partial^+_{n,m}, \quad E_{n,m} = \mathbb{E}_{V_{n,m}} \setminus \mathbb{E}_{\Lambda^c_{n,m}}.$$

We will use several mappings between models defined on $(\mathbb{L}_\bullet, \mathbb{E}^\bullet)$, $(\mathbb{L}_\circ, \mathbb{E}^\circ)$, or on the two simultaneously. A convenient way to encode all these mappings is to work with $\mathbb{L}_\diamond$, the set of *mid-points* of edges (primal or dual, as they give the same), and to identify the mid-edges with the corresponding tiles: to each primal-dual pair of edges e, e^*, associate a tile given by the convex hull of their endpoints; see Figure 2. For $t \in \mathbb{L}_\diamond$, denote e_t the associated primal edge, and e^*_t the associated dual edge.

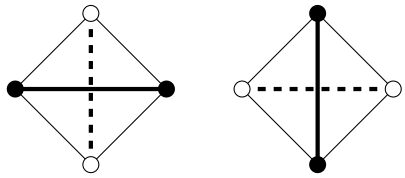

FIGURE 2. Tile associated to a mid-edge.

For a set of primal (or dual) edges E, define $*E = \{e^* : \ e \in E\}$. As a convention, sets of edges or of dual edges will be identified with the corresponding sets of mid-points whenever the meaning is clear from the context.

Also, for a set of mid-edges $A \subset \mathbb{L}_\diamond$, denote by $V_\bullet(A)$ the set of primal vertices belonging to a tile in A, and by $V_\circ(A)$ the set of dual vertices belonging to a tile in A. Let ∂A be the set of tiles in A adjacent to at least two tiles in A^c, and set $\mathring{A} = A \setminus \partial A$. For a set E of primal or dual edges, denote by $\mathcal{T}(E)$ the set of tiles having at least one corner in $\mathbb{V}_E$, and one corner in $\mathbb{V}_{*E}$.

2.2. **Parameters.** The parameter q will be fixed in all proofs and satisfy $q > 4$. When not mentioned explicitly, the parameter $\beta = \frac{1}{T}$ will also be fixed and set to $\beta = \beta_c(q) = \ln\big(1 + \sqrt{q}\big)$. In the same fashion, unless explicitly stated, the parameters U, J will also be fixed and satisfy $U \geq J \geq 0$. Additional constraints will be imposed in the concerned sections.

2.3. **Constants.** Constants like $c, c_1, C, C_1, C', \dots$ are constants which can change from line to line and which can depend on the parameters unless explicitly stated. They are independent of the system size, n, which will be our main "variable" quantity.

3. Ashkin–Teller random-cluster model

Like the Potts models, the AT model has a random-cluster (RC) representation, called the *ATRC model*, and itroduced in [CM97, PV97]. We will first introduce the model and state some of its basic properties. This will be followed by the statement of our results.

3.1. **Definition and basic properties.** Let $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ be a graph, and let $G = (V, E)$ be a finite sub-graph of $\mathbb{G}$. Let $\xi_\tau, \xi_{\tau\tau'} \in \{0,1\}^{\mathbb{E}}$ with $\xi_\tau \subseteq \xi_{\tau\tau'}$ The ATRC model on G with parameters $U > J > 0$ under boundary conditions $(\xi_\tau, \xi_{\tau\tau'})$ is the probability measure on $\{0,1\}^{\mathbb{E}} \times \{0,1\}^{\mathbb{E}}$ given by

$$\begin{aligned} \mathsf{ATRC}^{\xi_\tau,\xi_{\tau\tau'}}_{G;J,U}(\omega_\tau, \omega_{\tau\tau'}) = \tfrac{1}{Z} \cdot \mathrm{w}_\tau^{|\omega_\tau \cap E|} \, \mathrm{w}_{\tau\tau'}^{|(\omega_{\tau\tau'} \setminus \omega_\tau) \cap E|} \, 2^{\kappa_V(\omega_\tau) + \kappa_V(\omega_{\tau\tau'})} \\ \cdot \mathbb{1}_{\omega_\tau \subseteq \omega_{\tau\tau'}} \prod_{e \in \mathbb{E} \setminus E} \mathbb{1}_{\omega_\tau(e) = \xi_\tau(e)} \mathbb{1}_{\omega_{\tau\tau'}(e) = \xi_{\tau\tau'}(e)}, \end{aligned} \tag{3}$$

where $Z = Z^{\xi_\tau,\xi_{\tau\tau'}}_{\mathsf{ATRC}}(G, J, U)$ is the partition function, $\kappa_V(\cdot)$ is the number of clusters that intersect V, and the weights are given by

$$\mathrm{w}_\tau = e^{2U}(e^{2J} - e^{-2J}) \quad \text{and} \quad \mathrm{w}_{\tau\tau'} = e^{2(U-J)} - 1. \tag{4}$$

When $\xi_\tau \equiv 0$ (respectively, $\xi_\tau \equiv 1$), we write 0 (respectively, 1) instead of ξ_τ in the superscript, and analogously for $\xi_{\tau\tau'}$. Given finite subsets $\Lambda \subset \mathbb{V}$ and $E \subset \mathbb{E}$, we write $\mathsf{ATRC}^{\xi_\tau,\xi_{\tau\tau'}}_{\Lambda;J,U}$ and $\mathsf{ATRC}^{\xi_\tau,\xi_{\tau\tau'}}_{E;J,U}$ for the measures on the graphs $(\Lambda, \mathbb{E}_\Lambda)$ and $(\mathbb{V}_E, E)$, respectively.

Finite energy. There exists a constant $c = c(J, U) > 0$ such that the following holds. For any finite sub-graph $G = (V, E)$, any $e \in E$ and any $a, b \in \{0,1\}^E$ with $a \subseteq b$

$$\mathsf{ATRC}^{\xi_\tau,\xi_{\tau\tau'}}_{G;J,U}\big((\omega_\tau(e), \omega_{\tau\tau'}(e)) = (a_e, b_e) \,\big|\, (\omega_\tau, \omega_{\tau\tau'}) = (a, b) \text{ on } E \setminus \{e\}\big) > c. \tag{FE}$$

Spatial Markov property. Given a sub-graph $G' = (V', E')$ of G and boundary conditions $\xi_\tau, \xi_{\tau\tau'}, \tilde{\xi}_\tau, \tilde{\xi}_{\tau\tau'}$ with $\xi_\tau = \tilde{\xi}_\tau$ on $\mathbb{E} \setminus E'$ and $\xi_{\tau\tau'} = \tilde{\xi}_{\tau\tau'}$ on $\mathbb{E} \setminus E'$,

$$\mathsf{ATRC}^{\xi_\tau,\xi_{\tau\tau'}}_{G;J,U}(\,\cdot\,|\,(\omega_\tau, \omega_{\tau\tau'}) = (\tilde{\xi}_\tau, \tilde{\xi}_{\tau\tau'}) \text{ on } \mathbb{E} \setminus E') = \mathsf{ATRC}^{\tilde{\xi}_\tau,\tilde{\xi}_{\tau\tau'}}_{G';J,U}(\cdot). \tag{SMP}$$

Stochastic domination and positive association. We first introduce these notions in a general setting. Given a finite partially ordered set $\mathcal{S}$ and an index set I, we equip the set of functions $\mathcal{S}^I$ with the associated product order. A subset $A \subseteq \mathcal{S}^I$ is called *increasing* if for any $\omega, \omega' \in \mathcal{S}^I$, $\omega \in A$ and $\omega \leq \omega'$ implies $\omega' \in A$. Given a σ-algebra $\mathcal{A}$ on $\mathcal{S}^I$ and two probability measures μ_1 and μ_2 on $\mathcal{A}$, we say that μ_1 is stochastically dominated by μ_2 (or μ_2 stochastically dominates μ_1), and write $\mu_1 \leq_{\mathrm{st}} \mu_2$ (or $\mu_2 \geq_{\mathrm{st}} \mu_1$), if for every increasing event $A \in \mathcal{A}$, we have $\mu_1(A) \leq \mu_2(A)$. Moreover, we say that a probability measure μ on $\mathcal{A}$ is *positively associated* or satisfies the *FKG property* if, for all increasing events $A, B \in \mathcal{A}$, we have

$$\mu(A \cap B) \geq \mu(A)\mu(B). \tag{FKG}$$

If the index set I is finite and μ is a positive measure ($\mu(\omega) > 0$ for any $\omega \in \mathcal{S}^I$), we say that μ is *strongly positively associated* or satisfies the *strong FKG* property if

$$\forall\, J \subset I \;\forall \tilde{\omega} \in \mathcal{S}^J : \quad \mu(\,\cdot\,|\,\omega = \tilde{\omega} \text{ on } J) \quad \text{is positively associated.} \tag{strong-FKG}$$

In order to incorporate pairs of percolation configurations into the above general framework, we consider the natural bijection with $(\{0,1\} \times \{0,1\})^{\mathbb{E}}$. Furthermore, we can regard $\mathsf{ATRC}^{\xi_\tau,\xi_{\tau\tau'}}_{G;J,U}$ as a positive measure on $\{(0,0), (0,1), (1,1)\}^E$. The following lemma follows from [PV97, Proposition 4.1] (and its proof).

Lemma 3.1 ([PV97]). *For any $U > J > 0$, any sub-graph $G = (V, E)$ of $\mathbb{G}$ and all boundary conditions $\xi_\tau, \xi_{\tau\tau'}$, the measure $\mathsf{ATRC}^{\xi_\tau,\xi_{\tau\tau'}}_{G;J,U}$ satisfies the strong FKG property.*

The strong FKG property and (SMP) imply that, for any increasing sequence of subgraphs $G_k \nearrow \mathbb{G}$, the measures $\mathsf{ATRC}^{1,1}_{G_k;J,U}$ form a decreasing sequence (in the sense of stochastic domination). Consequently, the weak limit exists and is independent of the sequence (G_k), and we denote it by $\mathsf{ATRC}^{1,1}_{J,U}$. Analogously, we define $\mathsf{ATRC}^{0,0}_{J,U}$ as the (increasing) limit of $\mathsf{ATRC}^{0,0}_{G_k;J,U}$.

Furthermore, Lemma 3.1 and (SMP) can be used to compare different boundary conditions. For any $U > J > 0$, and any boundary conditions such that $\xi_\tau \leq \tilde{\xi}_\tau$ and $\xi_{\tau\tau'} \leq \tilde{\xi}_{\tau\tau'}$,

$$\mathsf{ATRC}^{\xi_\tau,\xi_{\tau\tau'}}_{G;J,U} \leq_{\mathrm{st}} \mathsf{ATRC}^{\tilde{\xi}_\tau,\tilde{\xi}_{\tau\tau'}}_{G;J,U}. \tag{CBC}$$

Edwards–Sokal relations on the square lattice. Let $\mathbb{G}$ be the square lattice $(\mathbb{L}_\bullet, \mathbb{E}^\bullet)$ defined in Section 2. Given $\Lambda, \Delta \subset \mathbb{L}_\bullet$ and a percolation configuration $\omega \in \{0,1\}^{\mathbb{E}^\bullet}$, we write $\Lambda \overset{\omega}{\leftrightarrow} \Delta$ for the event that Λ and Δ are connected by a path in the graph $(\mathbb{L}_\bullet, \omega)$. If $\Lambda = \{i\}$ and $\Delta = \{j\}$, we simply write $i \overset{\omega}{\leftrightarrow} j$. We omit ω from the notation when it cannot lead to any confusion.

Recall the definition of the infinite-volume AT expectation operators $\langle \cdot \rangle^{\mathrm{f,f}}$ and $\langle \cdot \rangle^{+,+}$ defined in Section 1.3. The essential feature of the ATRC is that its connection probabilities are related to the correlations in the AT model [PV97, Proposition 3.1]: for $U > J > 0$ and any $i, j \in \mathbb{L}_\bullet$,

$$\langle \tau_i \tau_j \rangle^{+,+} = \mathsf{ATRC}^{1,1}_{J,U}(i \overset{\omega_\tau}{\longleftrightarrow} j), \qquad \langle \tau_i \tau'_i \tau_j \tau'_j \rangle^{+,+} = \mathsf{ATRC}^{1,1}_{J,U}(i \overset{\omega_{\tau\tau'}}{\longleftrightarrow} j). \tag{5}$$

The analogous relations for $\langle \cdot \rangle^{\mathrm{f,f}}$ and $\mathsf{ATRC}^{0,0}_{J,U}$ are also valid.

Duality on the square lattice. Recall the definitions of the primal and dual square lattices $(\mathbb{L}_\bullet, \mathbb{E}^\bullet)$ and $(\mathbb{L}_\circ, \mathbb{E}^\circ)$ in Section 2, and the definition of the dual $\omega^* \in \{0,1\}^{\mathbb{E}^\circ}$ of a percolation configuration $\omega \in \{0,1\}^{\mathbb{E}^\bullet}$ in Section 1. For any finite subset $E \subset \mathbb{E}^\bullet$ and all boundary conditions $\xi_\tau, \xi_{\tau\tau'} \in \{0,1\}^{\mathbb{E}^\bullet}$, it holds that

$$(\omega_\tau, \omega_{\tau\tau'}) \sim \mathsf{ATRC}^{\xi_\tau,\xi_{\tau\tau'}}_{E;J,U} \quad \text{implies} \quad (\omega^*_{\tau\tau'}, \omega^*_\tau) \sim \mathsf{ATRC}^{\xi^*_\tau,\xi^*_{\tau\tau'}}_{*E;J,U}. \tag{6}$$

Notice the different order in the dual pair.

3.2. **Results on the ATRC.** In this section, we present our main results concerning the self-dual ATRC model on the square lattice $\mathbb{L}_\bullet$.

Mixing properties. Our first result regarding the mixing behaviour of the ATRC is exponential relaxation for the single edges, consequently implying *exponential weak mixing* (see Section 5).

Proposition 3.2. *Let $0 < J < U$ satisfy $\sinh 2J = e^{-2U}$. There exists $c > 0$ such that, for any edge $e \in \mathbb{E}^\bullet$ with $0 \in e$, and any $n \geq 1$,*

$$\max_{\sigma \in \{\tau, \tau\tau'\}} \left| \mathsf{ATRC}^{1,1}_{\Lambda_n;J,U}\big(\omega_\sigma(e) = 1\big) - \mathsf{ATRC}^{0,0}_{\Lambda_n;J,U}\big(\omega_\sigma(e) = 1\big) \right| \leq e^{-cn}.$$

As a consequence, the measures $\mathsf{ATRC}^{\xi_\tau,\xi_{\tau\tau'}}_{G;J,U}$ converge (as $G \nearrow \mathbb{L}_\bullet$) to a limit measure $\mathsf{ATRC}_{J,U}$, which is independent of the choice of boundary conditions $\xi_\tau, \xi_{\tau\tau'}$. Furthermore, the limit $\mathsf{ATRC}_{J,U}$ is the unique ATRC *Gibbs* measure.

We then apply the classical work of Alexander [Ale98] to derive *ratio* weak mixing.

Theorem 4. *Let $0 < J < U$ satisfy $\sinh 2J = e^{-2U}$. There exist $C \geq 0, c > 0$ such that, for any finite sub-graph $G = (V, E)$ of $\mathbb{L}_\bullet$, any $F \subset E$, and any F-measurable event A having positive probability,*

$$\sup_{\xi_\tau, \xi_{\tau\tau'}, \xi'_\tau, \xi'_{\tau\tau'}} \left| \frac{\mathsf{ATRC}^{\xi_\tau, \xi_{\tau\tau'}}_{G;J,U}(A)}{\mathsf{ATRC}^{\xi'_\tau, \xi'_{\tau\tau'}}_{G;J,U}(A)} - 1 \right| \leq C \sum_{f \in F} \sum_{e \in E^c} e^{-c\, \mathrm{d}_\infty(e,f)},$$

whenever the right side is strictly less than 1.

We refer to Section 6, for our results on *strong* mixing properties of the ATRC.

Uniform exponential decay. We say that a subset E of $\mathbb{E}^\bullet$ is *simply lattice-connected* if both $(\mathbb{V}_E, E)$ and the planar dual of $(\mathbb{V}_{E^c}, E^c)$ are connected. The following result is a consequence of [Ale04], exponential decay of connection probabilities in ω_τ [ADG24, Proposition 1.1], and the exponential weak mixing property (Theorem 7).

Theorem 5. *There exists a constant $c > 0$ such that, for any finite simply lattice-connected $E \subset \mathbb{E}$, and any $i, j \in \mathbb{V}_E$,*

$$\mathsf{ATRC}^{1,1}_{E;J,U}\big(i \xleftrightarrow{\omega_\tau \cap E} j\big) \leq e^{-c\, \mathrm{d}_\infty(i,j)},$$

where d_∞ is the distance induced by the L^∞ norm.

Proof. This is [Ale04, Theorem 1.1]: the push-forward of $\mathsf{ATRC}_{J,U}$ by ω_τ is translation invariant, has finite-energy for closing edges, exponential decay of connectivities [ADG24, Proposition 1.1], and is exponentially weak mixing by Theorem 7. Finally, for n with $E \subseteq \mathbb{E}_{\Lambda_{n-1}}$, we have by (SMP) that

$$\mathsf{ATRC}_{J,U}(\omega_\tau \in \cdot \,|\, \omega_\tau(e) = 1 \text{ for } e \in \mathbb{E}_{\Lambda_n} \setminus E) = \mathsf{ATRC}^{1,1}_{E;J,U}.$$

□

Ornstein–Zernike asymptotics. Recall the definition of the inverse correlation length ν in the AT model. By the Edwards–Sokal relations (5) and since $\mathsf{ATRC}^{0,0}_{J,U} = \mathsf{ATRC}^{1,1}_{J,U}$, it holds that

$$\nu(x) = -\lim_{n\to\infty} \frac{1}{n} \log \mathsf{ATRC}^{0,0}_{J,U}(0 \xleftrightarrow{\omega_\tau} \lfloor nx \rfloor) = -\lim_{n\to\infty} \frac{1}{n} \log \mathsf{ATRC}^{1,1}_{J,U}(0 \xleftrightarrow{\omega_\tau} \lfloor nx \rfloor).$$

The following theorem is the analogue of Theorem 3 for the ATRC model.

Theorem 6. *The inverse correlation length ν is a norm on $\mathbb{R}^2$. Furthermore, uniformly in $|x| \to \infty$,*

$$\mathsf{ATRC}^{0,0}_{J,U}(0 \xleftrightarrow{\omega_\tau} x) = \mathsf{ATRC}^{1,1}_{J,U}(0 \xleftrightarrow{\omega_\tau} x) = \frac{\Psi(x/|x|)}{\sqrt{|x|}}\, e^{-\nu(x)}\, (1 + o(1)),$$

where Ψ is a strictly positive analytic function on $\mathbb{S}^1$.

4. Couplings and interfaces

This section is concerned with the construction of a coupling of FK percolation and the Ashkin–Teller model, via the six-vertex model, and the derivation of its basic properties. The coupling of the FK and six-vertex measures is a version of the Baxter–Kelland–Wu (BKW) coupling [BKW76]. The relation of the six-vertex and AT measures has first been noticed in [Fan72a] comparing their critical properties, and it was made explicit in [Fan72b, Weg72] on a level of partition functions. We build on [GP23], where a coupling of the six-vertex model and a graphical representation of the AT model (a marginal of ATRC) was constructed.

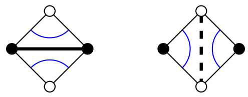

FIGURE 3. A tile centred at the middle of a horizontal primal edge (solid black) or its associated vertical dual edge (dashed black), and its two possible local loop configurations.

4.1. **The parameters.** The couplings go through standard "expansion–resummation" of Boltzmann weights combined with extensive use of planarity. As for each step one will write the weight associated with a given model as a sum of weights for a "more expanded" model, several parameters will come into play. We list them here, as well as the algebraic relations linking them:

$$q > 4, \quad \beta = \beta_c(q) = \ln(1+\sqrt{q}), \quad \lambda > 0, \quad \mathbf{c} > 2, \quad U > J > 0,$$
$$\sqrt{q} = e^{\lambda} + e^{-\lambda}, \quad \mathbf{c} = e^{\lambda/2} + e^{-\lambda/2} = \coth(2J), \tag{7}$$
$$\sinh(2J) = e^{-2U}.$$

The parameter $\mathbf{c}$ also has a "boundary version":

$$\mathbf{c}_{\mathrm{b}} > 1, \quad \mathbf{c}_{\mathrm{b}} = e^{\lambda/2}. \tag{8}$$

For the remainder of Section 4, we fix $q > 4$, along with the corresponding parameters above (which are uniquely determined by q).

4.2. **Different models and combinatorial mappings.** This section provides an overview of the combinatorial objects that will be encountered, as well as a description of their relations. We first discuss oriented loop configuration, which serve as an intermediate step in the BKW coupling of FK percolation and the six-vertex model. This is followed by a description of two of the representations of the six-vertex model: edge-orientations and spin configurations. The height function representation appears only in Section 5, where its monotonicity properties are crucial when we derive mixing and relaxation properties of the ATRC measures.

We already saw that percolation configurations are in bijection with (unoriented) loop configurations. To make the correspondence $\omega \leftrightarrow \omega^* \leftrightarrow \ell = \mathrm{loop}(\omega)$ explicit, we can regard each of these models as an assignment of a local piece of drawing of an edge and two *arcs* to lozenge tiles centred at the mid-edges as depicted in Figure 3. Clearly, retaining only either the primal or dual edges, or the arcs, provides complete information about all three.

Oriented loop configurations. They are obtained from unoriented loop configurations by assigning an orientation to each loop [BKW76], or equivalently by assigning orientations to the loop arcs on each tile, subject to the constraint that neighbouring orientations match. The eight local configurations that can occur at a tile are referred to as *types*, see Fig. 4. Boundary conditions with respect to a tile-set $A \subset \mathbb{L}_\diamond$ are imposed by conditioning the tiles in $\mathbb{L}_\diamond \setminus A$ to take prescribed values, and by forcing the tiles in ∂A to contain a given oriented arc.

The edge-orientations of the six-vertex model. The *medial graph* of $\mathbb{L}_\bullet$ has vertex-set $\mathbb{L}_\diamond$ and edges between adjacent tiles. We denote its edge-set by $\mathbb{E}^\diamond$. The edge-orientations of the six-vertex model [Pau35, Rys63] are assignments of orientations to the edges in $\mathbb{E}^\diamond$, obtained from oriented loop configurations via the natural surjection; see Fig. 4. The edge orientations that we obtain in this way satisfy the *ice rule*: at any

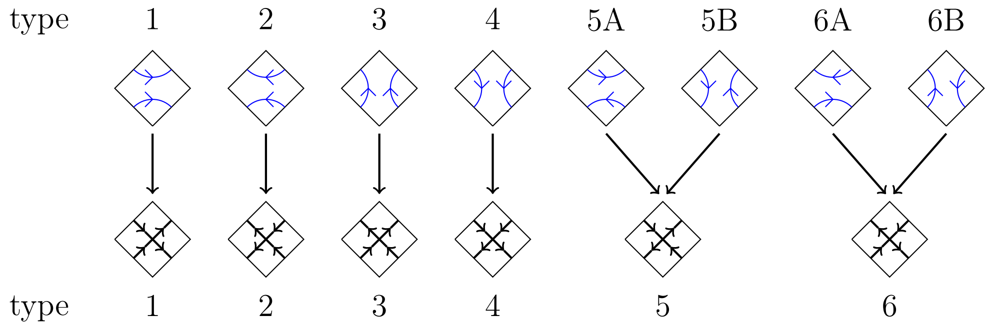

FIGURE 4. Tiles of the oriented loop model and their types and weights, and the mapping from oriented loop arcs to six-vertex edge-orientations.

mid-edge in $\mathbb{L}_\diamond$, there are two incoming and two outgoing edges of $\mathbb{E}^\diamond$. This constraint permits six possible local configurations at a tile, which are also called types; see Fig. 4. The local inverse operation can be considered as *splitting* the oriented edges into two oriented loop arcs. While tiles of types 1-4 permit a unique reconstruction of the loop arcs based on the edge orientations, there are two possibilities for tiles of types 5-6, giving the latter a special role.

The six-vertex spin representation. The six-vertex model may be represented by pairs of spin-configurations $(\sigma_\bullet, \sigma_\circ) \in \{\pm 1\}^{\mathbb{L}_\bullet} \times \{\pm 1\}^{\mathbb{L}_\circ}$ [Wu71, KW71, Lis22, GP23], obtained from the edge-orientations via the following two-valued mapping. Fix the value of $\sigma_\bullet$ or $\sigma_\circ$ at some arbitrary fixed vertex and proceed iteratively as follows. Take an edge $e \in \mathbb{E}^\diamond$ and observe that it separates a vertex $i \in \mathbb{L}_\bullet$ from a vertex $u \in \mathbb{L}_\circ$. The values of $\sigma_\bullet(i)$ and $\sigma_\circ(u)$ are defined in such a way that i is positioned on the left side of e (with respect to its assigned orientation) precisely if $\sigma_\bullet(i) = \sigma_\circ(u)$; see Fig. 5. The ice rule ensures that this operation is well-defined.

This mapping is two-valued due to the liberty to choose the value of $\sigma_\bullet$ or $\sigma_\circ$ at a fixed vertex, and the two images are related to each other by a global spin flip. Furthermore, it is injective, meaning that the edge-orientations can be reconstructed from the spins. The type of a tile with respect to $(\sigma_\bullet, \sigma_\circ)$ is given by the type of the corresponding edge-orientations; see Fig. 5. It should also be noted that the ice rule can be translated as follows: for any tile $t \in \mathbb{L}_\diamond$, $\sigma_\bullet$ is constant on the endpoints of e_t or $\sigma_\circ$ is constant on the endpoints of e_t^*. Formally,

$$\big(\sigma_\bullet(i) - \sigma_\bullet(j)\big)\big(\sigma_\circ(u) - \sigma_\circ(v)\big) = 0 \qquad \text{for any } t \in \mathbb{L}_\diamond \text{ with } e_t = ij,\ e_t^* = uv. \tag{9}$$

In the context of spins, this property will henceforth be referred to as the ice rule.

Baxter–Kelland–Wu correspondence [BKW76]. Given a measure on oriented loops, taking its pushforwards with respect to the above mappings, one obtains measures on the six-vertex edge-orientations and spin configurations, the latter necessarily having a prescribed value at a fixed vertex. In order to obtain an oriented loop configuration $\ell_\rightarrow$ from an unoriented one ℓ, we will utilise a sequence of independent uniform random variables on $[0, 1]$, indexed by the set $\mathcal{L}$ of all loops, to randomly assign an orientation to each loop l in ℓ.

4.3. **Coupling under Dobrushin conditions.** The idea is to take the FK measure with Dobrushin boundary conditions, and from it construct via coupling both a measure on pairs of spin configurations on $\mathbb{L}_\bullet$ and $\mathbb{L}_\circ$ and a measure on ATRC configurations. We will then identify the measure on spin configurations as a six-vertex spin

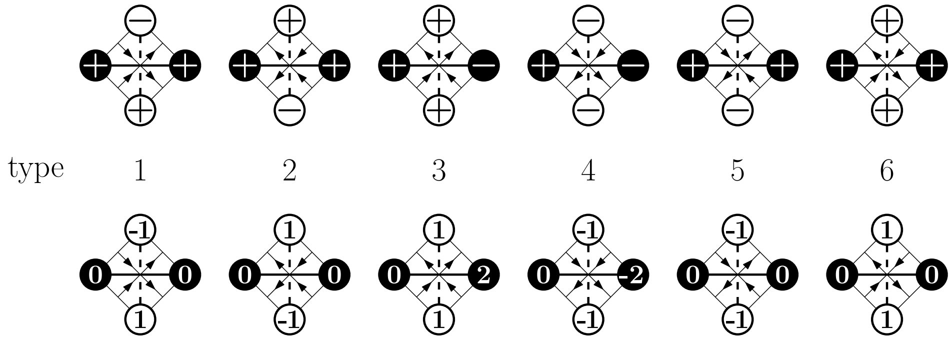

FIGURE 5. The six-vertex types for all representations at a tile corresponding to a horizontal primal edge e. Top: the spin at the left endpoint of e is fixed to be $+$. Bottom: the height at the left endpoint of e is fixed to be 0.

measure with Dobrushin boundary conditions, and show that the measure on ATRC configurations is a modification of an ATRC measure. This procedure gives a coupling between FK and (a modification of) ATRC, which will allow us to transfer the study of the former to the study of the latter.

Recall the sub-graphs $G_{n,m} = (V_{n,m}, E_{n,m})$ of $(\mathbb{L}_\bullet, \mathbb{E}^\bullet)$ given in Section 2. As n, m will be fixed in this section, we will omit them in the notation and simply write $G = (V, E)$. The FK measure under investigation is $\mathsf{FK}_G^{\eta_{1/0}} \equiv \mathsf{FK}_G^{1/0}$ (with $\eta_{1/0}$ the Dobrushin boundary conditions defined in Section 1; see Fig. 1).

Coupling measure. The couplings will use additional randomness in the form of i.i.d. uniform $[0,1]$ random variables. We will work with the space $\Omega^{1/0} := \{0,1\}^{\mathbb{E}^\bullet} \times [0,1]^{\mathcal{L}} \times [0,1]^{\mathbb{L}_\diamond}$, where $\mathcal{L}$ is the set of all unoriented loops . We consider a uniform measure $[0,1]$ equipped with the Borel sigma-algebra and define Q and Q' respectively as the product measures on $[0,1]^{\mathcal{L}}$ and on $[0,1]^{\mathbb{L}_\diamond}$ equipped with the product sigma-algebra. Take (ω, U, U') distributed according to $\Psi^{1/0} := \mathsf{FK}_G^{1/0} \otimes Q \otimes Q'$. We will use this augmented space to couple $\mathsf{FK}_G^{1/0}$ with both a six-vertex spin measure and a modified version of the ATRC measure.

From FK to six-vertex. We will employ the link between the unoriented and oriented loop models and the six-vertex spin model: the six-vertex spin configurations are obtained from the edge-orientations by fixing the value at one vertex (see Fig. 5). The edge orientations are obtained from oriented loop configurations by the natural map (see Fig. 4), whence it suffices to construct a measure on oriented lopp configurations. This is achieved by modifying the BKW coupling [BKW76]. In Section 5.1.4, we will describe this coupling without Dobrushin boundary conditions and in the context of height functions.

Recall that percolation configurations are in bijection with unoriented loop configurations, and let ℓ be the (random) unoriented loop configuration associated with $\omega \sim \mathsf{FK}_G^{1/0}$. We first define the relevant sets (see Fig.6):

- the interior of V in the graph $(\mathbb{L}_\bullet, \eta_{1/0})$, $\Lambda := \Lambda_{n,m} \subset \mathbb{L}_\bullet$;
- the interior of $\mathbb{V}_{*E}$ in the graph $(\mathbb{L}_\circ, \eta^*_{1/0})$,

$$\Lambda' := \Big(\big([-n-1, n+1] \times [0, m+1] \big) \cup \big([-n, n] \times [-m, 0] \big) \Big) \cap \mathbb{L}_\circ.$$

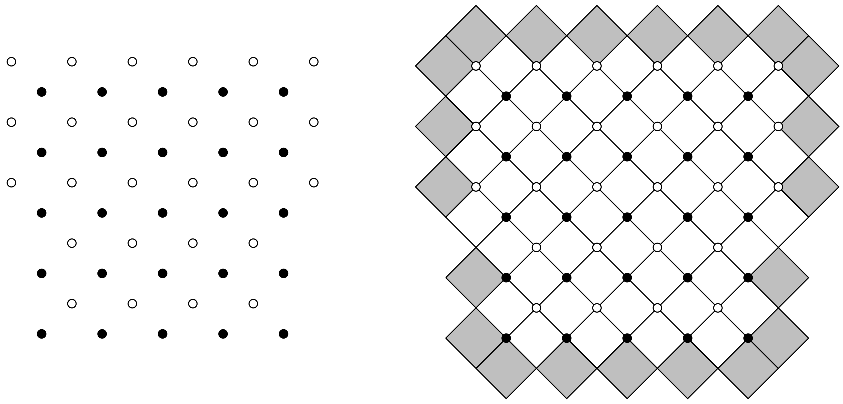

FIGURE 6. For $n = m = 2$. Left: the sets Λ (solid) and Λ' (hollow). Right: the inner tiles A^{i} (white) and the boundary tiles ∂A (grey).

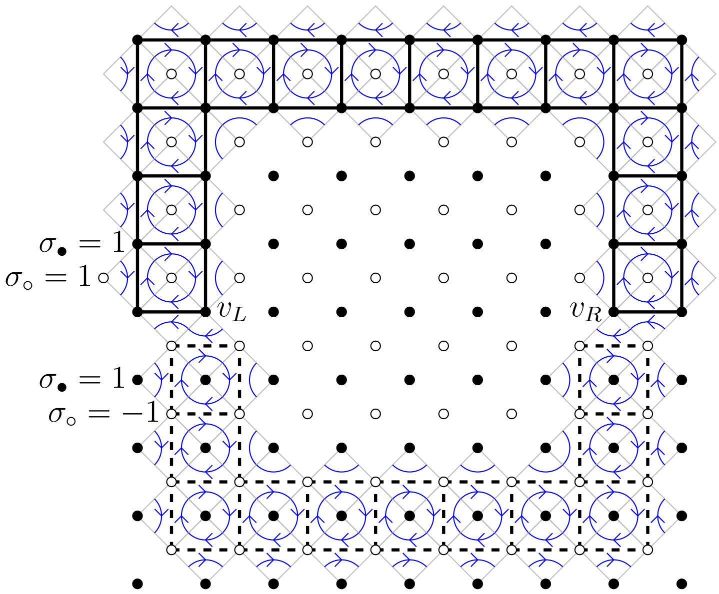

FIGURE 7. Boundary conditions on oriented loops and on six-vertex spins. The edges drawn are those induced by the oriented loop boundary conditions.

Define $\mathcal{D} := \Lambda \cup \Lambda'$, and let $A \subset \mathbb{L}_\diamond$ be given by the tiles with at least one corner in $\mathcal{D}$. The set ∂A of boundary tiles is given by the tiles in A with precisely one corner in $\mathcal{D}$, and the set A^{i} of inner tiles is given by $A \setminus \partial A$. See again Figure 6.

By orienting the loops in ℓ, we will construct an oriented loop configuration $\ell_\rightarrow$. We will first define its boundary conditions; see Fig. 7: each (dual) vertex in $(\mathbb{L}_\circ \cap \mathbb{H}^+) \setminus \Lambda'$ and each (primal) vertex in $(\mathbb{L}_\bullet \cap \mathbb{H}^-) \setminus \Lambda$ is surrounded by a clockwise oriented loop consisting of four arcs, and there is a bi-infinite right-left path in between.

It remains to define the oriented loop arcs of $\ell_\rightarrow$ on tiles in A^{i}. Recall the definition of λ and $\mathbf{c}$ in (7). Take an unoriented loop l in ℓ that surrounds a vertex in $\Lambda \cup \Lambda'$, and orient it clockwise if $U_l < e^\lambda/\mathbf{c}$ and counter-clockwise otherwise. Finally, consider the six-vertex edge orientations obtained from $\ell_\rightarrow$, and let $(\sigma_\bullet, \sigma_\circ)$ be the associated six-vertex spin configurations with $\sigma_\bullet((n+1, 0)) = +1$. Observe that the boundary conditions of the edge orientations imposed by those of the oriented loops $\ell_\rightarrow$ (see

Fig. 7) impose the following boundary conditions on $(\sigma_\bullet, \sigma_\circ)$:

$$\sigma_\bullet \in \Sigma_\Lambda^+ := \{\sigma_\bullet \in \{\pm 1\}^{\mathbb{L}_\bullet} : \sigma_\bullet(i) = 1 \, \forall i \in \mathbb{L}_\bullet \setminus \Lambda\},$$
$$\sigma_\circ \in \Sigma_{\Lambda'}^{+-} := \{\sigma_\circ \in \{\pm 1\}^{\mathbb{L}_\circ} : \sigma_\circ(u) = \mathbb{1}_{\mathbb{H}^+}(u) - \mathbb{1}_{\mathbb{H}^-}(u) \, \forall u \in \mathbb{L}_\circ \setminus \Lambda'\}.$$

Note that the vertices in $\mathbb{L}_\circ \setminus \Lambda'$ where $\sigma_\circ$ takes opposite values are separated by the right-left path. The following lemma will identify the law of $(\sigma_\bullet, \sigma_\circ)$ as a six-vertex spin measure, which we will define first. To lighten notation, we will use the same symbols for random variables and their deterministic realisations. Define the probability measure $\mathsf{Spin}_{\mathcal{D}}^{+,+-}$ on $\{\pm 1\}^{\mathbb{L}_\bullet} \times \{\pm 1\}^{\mathbb{L}_\circ}$ by setting, for $\sigma = (\sigma_\bullet, \sigma_\circ)$,

$$\mathsf{Spin}_{\mathcal{D}}^{+,+-}(\sigma) \propto \mathbf{c}^{|T_{5,6}^{\mathrm{i}}(\sigma)|} \, \mathbf{c}_{\mathrm{b}}^{|T_{5,6}^{\mathrm{b}}(\sigma)|} \, \mathbb{1}_{\Sigma_\Lambda^+ \times \Sigma_{\Lambda'}^{+-}}(\sigma) \, \mathbb{1}_{\mathrm{ice}}(\sigma), \tag{10}$$

where $\mathbf{c}, \mathbf{c}_{\mathrm{b}}$ are defined by (7)-(8), $T_{5,6}^{\mathrm{i}}$ and $T_{5,6}^{\mathrm{b}}$ are the sets of tiles of types 5-6 in A^{i} and ∂A, respectively, according to Figure 5, and $\mathbb{1}_{\mathrm{ice}}$ is the indicator imposing the ice rule (9).

Lemma 4.1. *Let (ω, U_1, U_2) be distributed according to $\Psi^{1/0}$, and let $(\sigma_\bullet, \sigma_\circ)$ be as constructed above. Then the law of $(\sigma_\bullet, \sigma_\circ)$ is given by $\mathsf{Spin}_{\mathcal{D}}^{+,+-}$.*

Proof. We follow the ideas of [BKW76]. One has to examine which values of (ω, U) give a fixed $(\sigma_\bullet, \sigma_\circ) \in \{\pm 1\}^{\mathbb{L}_\bullet} \times \{\pm 1\}^{\mathbb{L}_\circ}$. The probability to obtain $(\sigma_\bullet, \sigma_\circ) \in \Sigma_\Lambda^+ \times \Sigma_{\Lambda'}^{+-}$ is the probability to obtain its associated six-vertex edge orientations. This is therefore the sum of the probabilities of all oriented loop configurations $\ell_\rightarrow$ that induce these edge orientations. The probability of a given oriented loop configuration $\ell_\rightarrow$ satisyfing the boundary conditions in Fig. 7 is proportional to

$$\sqrt{q}^{\,|\mathrm{loop}(\ell_\rightarrow)|} \cdot \left(\tfrac{e^\lambda}{e^\lambda + e^{-\lambda}}\right)^{|\mathrm{loop}_{\circlearrowright}(\ell_\rightarrow)| - |\mathrm{loop}_{\circlearrowleft}(\ell_\rightarrow)|} = e^{\lambda(|\mathrm{loop}_{\circlearrowright}(\ell_\rightarrow)| - |\mathrm{loop}_{\circlearrowleft}(\ell_\rightarrow)|)}, \tag{11}$$

where $\mathrm{loop}_{\circlearrowright}(\ell_\rightarrow)$ and $\mathrm{loop}_{\circlearrowleft}(\ell_\rightarrow)$ are respectively the sets of clockwise and counter-clockwise oriented loops in $\ell_\rightarrow$ not imposed by boundary conditions, and $\mathrm{loop}(\ell_\rightarrow)$ is their union. Indeed, by Lemma 1, the first factor on the right side is proportional to $\mathsf{FK}_G^{1/0}(\omega(\ell))$, where $\omega(\ell) \in \{0,1\}^{\mathbb{E}^\bullet}$ is the percolation configuration associated to the unoriented loop configuration ℓ corresponding to $\ell_\rightarrow$. The second factor comes from the values of the uniforms necessary to obtain correct orientations of loops in $\mathrm{loop}(\ell_\rightarrow)$. The equality holds since $\sqrt{q} = e^\lambda + e^{-\lambda}$ due to the choice of λ.

Notice that each loop which is oriented clockwise does 4 more right quarter-turns than left quarter-turns, that the converse holds for counter-clockwise oriented loops, and that the number of left and right quarter-turns of the crossing path differ by a universal constant. Consequently, the expression on the right side of (11) equals

$$\exp(\lambda(\#_{\curvearrowright}(\ell_\rightarrow) - \#_{\curvearrowleft}(\ell_\rightarrow))/4),$$

where $\#_{\curvearrowright}(\ell_\rightarrow)$ and $\#_{\curvearrowleft}(\ell_\rightarrow)$ are respectively the number of right and left quarter-turns in $\ell_\rightarrow$ that are not imposed by the boundary conditions. The key idea is to count these oriented loop arcs locally at each tile in $A = A^{\mathrm{i}} \cup A^{\mathrm{b}}$. Observe that, for tiles in A^{i}, types 5B,6A correspond to a pair of right-oriented loop arcs and types 5A,6B to a pair of left-oriented loop arcs, whereas types 1-4 correspond to one right-oriented and one left-oriented loop arc each. Moreover, due to the boundary conditions (see Fig. 7), a tile in A^{b} contains a right turn precisely if it is of type 5,6, and it contains a left turn otherwise. We deduce that the probability of $\ell_\to$ is proportional to

$$\Bigg(\prod_{t\in T_{5,6}(\ell_\to)\cap A^{\mathrm{i}}} e^{\lambda/2}\,\mathbb{1}_{T_{5B,6A}(\ell_\to)}(t)+e^{-\lambda/2}\,\mathbb{1}_{T_{5A,6B}(\ell_\to)}(t)\Bigg)\cdot\big(e^{\lambda/2}\big)^{|T_{5,6}(\ell_\to)\cap\partial A|},\tag{12}$$

where $T_{5B,6A}(\ell_\to)$ and $T_{5A,6B}(\ell_\to)$ are respectively the sets of tiles of types 5B and 6A and the set of tiles of types 5A and 6B in $\ell_\to$, and $T_{5,6}(\ell_\to)$ is their union.

Finally, fix a pair $(\sigma_\bullet,\sigma_\circ)\in\Sigma_\Lambda^+\times\Sigma_{\Lambda'}^{+-}$ that satisfies the ice-rule, and consider its associated edge-orientations. It remains to identify all oriented loop configurations $\ell_\to$ that satisfy the boundary conditions in Fig. 7 and that induce these edge-orientations. Observe that the boundary conditions and the spins $(\sigma_\bullet,\sigma_\circ)$ uniquely determine the oriented loop arcs at tiles in $(\mathbb{L}_\diamond\setminus A)\cup A^{\mathrm{b}}$ and at tiles in $A^{\mathrm{i}}\setminus T^{\mathrm{i}}_{5,6}(\sigma_\bullet,\sigma_\circ)$. For a tile in $T^{\mathrm{i}}_{5,6}(\sigma_\bullet,\sigma_\circ)$, one can split the oriented edges either into a pair of right-oriented loops arcs (types 5B,6A) or into a pair of left-oriented loop arcs (types 5A,6B). Summing the probabilities (12) of all oriented loop configurations obtained in that way, we obtain that the probability of $(\sigma_\bullet,\sigma_\circ)$ is proportional to

$$\big(e^{\lambda/2}+e^{-\lambda/2}\big)^{|T^{\mathrm{i}}_{5,6}(\sigma_\bullet,\sigma_\circ)|}\cdot\big(e^{\lambda/2}\big)^{|T^{\mathrm{b}}_{5,6}(\sigma_\bullet,\sigma_\circ)|}.$$

Recalling that $\mathbf{c}=e^{\lambda/2}+e^{-\lambda/2}$ and $\mathbf{c}_{\mathrm{b}}=e^{\lambda/2}$ finishes the proof. □

From six-vertex to modified ATRC. In line with the approach outlined in [GP23, Section 7], we make appropriate adjustments. Recall the notation introduced at the beginning of the subsection: the sets of primal and dual vertices Λ and Λ', $\mathcal{D}=\Lambda\cup\Lambda'$ and the set of tiles A that intersect $\mathcal{D}$. Let $E_1:=\{e_t : t\in A\}\supset E$, and define a pair of percolation configurations $\xi_\tau,\xi_{\tau\tau'}\in\{0,1\}^{E_1}$ as follows. For each tile $t\in A$, denoting $e_t=ij\in E_1$ and $e_t^*=uv$,

- if $\sigma_\circ(u)\neq\sigma_\circ(v)$, set $\xi_\tau(e_t)=\xi_{\tau\tau'}(e_t)=1$ (never happens for $t\in\partial A\cap\mathbb{H}^-$);
- if $\sigma_\bullet(i)\neq\sigma_\bullet(j)$, set $\xi_\tau(e_t)=\xi_{\tau\tau'}(e_t)=0$ (never happens for $t\in\partial A\cap\mathbb{H}^+$);
- if both $\sigma_\circ(u)=\sigma_\circ(v)$ and $\sigma_\bullet(i)=\sigma_\bullet(j)$ hold (types 5-6), let

$$(\xi_\tau(e_t),\xi_{\tau\tau'}(e_t))=\begin{cases}\mathbb{1}_{[0,\frac{1}{\mathbf{c}})}(U_t)\cdot(1,1)+\mathbb{1}_{[\frac{1}{\mathbf{c}},\frac{2}{\mathbf{c}})}(U_t)\cdot(0,0)+\mathbb{1}_{[\frac{2}{\mathbf{c}},1]}(U_t)\cdot(0,1) & \text{if } t\in A^{\mathrm{i}},\\ \mathbb{1}_{[0,\frac{1}{\mathbf{c}_{\mathrm{b}}})}(U_t)\cdot(1,1)+\mathbb{1}_{[\frac{1}{\mathbf{c}_{\mathrm{b}}},1]}(U_t)\cdot(0,0) & \text{if } t\in\partial A.\end{cases}$$

In particular, it holds that $\xi_\tau\subseteq\xi_{\tau\tau'}$ and $\xi_{\tau\tau'}\setminus\xi_\tau\subseteq\{e_t : t\in A^{\mathrm{i}}\}=E$.

For an edge $ij\in\mathbb{E}^\bullet$, we write $\sigma_\bullet\sim e$ if $\sigma_\bullet(i)=\sigma_\bullet(j)$; for $\xi\subset\mathbb{E}^\bullet$, we write $\sigma_\bullet\sim\xi$ if $\sigma_\bullet\sim e$ for every $e\in\xi$ (in other words, $\sigma_\bullet$ is constant on clusters of ξ). We use similar notation for $\mathbb{E}^\circ$ and $\sigma_\circ$. The above definition implies that $\sigma_\bullet\sim\xi_{\tau\tau'}$ and $\sigma_\circ\sim\xi_\tau^*$.

Remark 4.2. *The above sampling rule for $t\in A^{\mathrm{i}}$ applied to a six-vertex spin or height measure without modified boundary weight $\mathbf{c}_{\mathrm{b}}$ yields an ATRC measure, as defined in Secton 3. See Section 5.1.3 for details.*

Before providing an explicit expression for the law of $(\xi_\tau,\xi_{\tau\tau'})$ in the next lemma, we require some notation.

- Let K be the graph $(\mathbb{V}_{E_1},E_1)$, and let $K_\sim$ be the graph obtained from K by identifying the vertices in $\partial^{\mathrm{in}}_{\mathbb{L}_\bullet}\mathbb{V}_{E_1}$ (it has many self-edges). See Figure 8 (left).
- Let K' be the graph induced by edges dual to edges in E_1: $K'=(\mathbb{V}_{*E_1},*E_1)$, and let $K'_\sim$ be the graph obtained from K' by identifying the vertices in $\partial^{\mathrm{in}}_{\mathbb{L}_\circ}\mathbb{V}_{*E_1}$. See Figure 8.

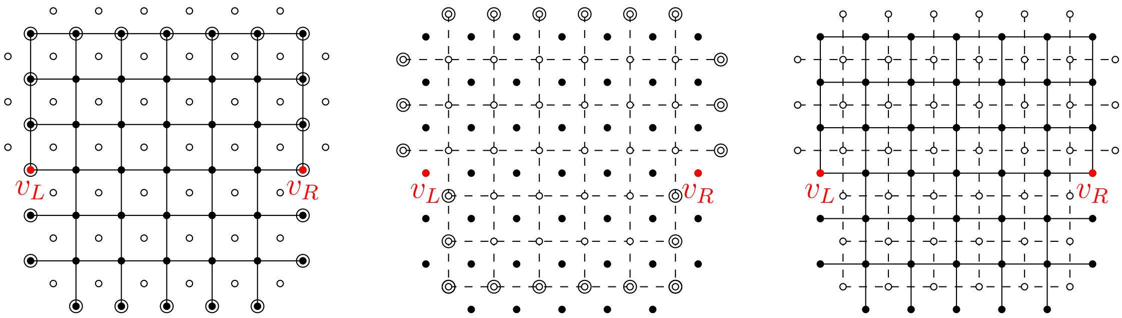

FIGURE 8. The graphs K (left), K' (center), and the planar duality relation between their edges (right). Vertices surrounded by a circle are identified in $K_\sim$ and $K'_\sim$, respectively.

One then has that the planar dual of K is $K'_\sim$, and the planar dual of K' is $K_\sim$.

Recall that n is the width of $\Lambda = \Lambda_{n,m}$, and define

$$v_L = (-n-1, 0), \quad v_R = (n+1, 0),$$

see Figures 7 and 8. Moreover, define $E_\mathrm{b} := \{e_t : t \in \partial A\} = E_1 \setminus E$, and set

$$E_\mathrm{b}^+ := \{e \in E_\mathrm{b} : e \subset \mathbb{H}^+\} \quad \text{and} \quad E_\mathrm{b}^- := E_\mathrm{b} \setminus E_\mathrm{b}^+.$$

The next lemma expresses the law of $(\xi_\tau, \xi_{\tau\tau'})$ as a certain measure on $\{0,1\}^{E_1} \times \{0,1\}^{E_1}$ conditioned on the event $v_L \overset{\xi_\tau}{\longleftrightarrow} v_R$. We first introduce this measure.

Definition 1. In the current definition, we view each $a \in \{0,1\}^{E_1}$ as a spanning subgraph of K. We say that a cluster of a is an *inner cluster* if it is entirely contained in Λ and a *boundary cluster* otherwise. Define $\kappa_K(a)$ as the number of clusters in ξ, and let $\mathrm{cl}_\Lambda(a)$ be the set of clusters in ξ that intersect Λ. The modified ATRC probability measure $\mathsf{mATRC}_{n,m} \equiv \mathsf{mATRC}_K$ on $\{0,1\}^{E_1} \times \{0,1\}^{E_1}$ is defined by

$$\mathsf{mATRC}_K(a,b) \propto \mathbb{1}_{a\subseteq b}\mathbb{1}_{b\setminus a\subseteq E} 2^{\kappa_K(a)} 2^{|a|} (\mathbf{c}-2)^{|b\setminus a|} (\mathbf{c}_\mathrm{b}-1)^{|E_\mathrm{b}^+\setminus b|} \cdot \prod_{\mathcal{C}\in \mathrm{cl}_\Lambda(b)} \big(\mathbb{1}_{\mathcal{C}\subseteq\Lambda} + (\mathbf{c}_\mathrm{b}-1)^{|E_\mathrm{b}^-\cap\partial^{\mathrm{edge}}\mathcal{C}|}\big).$$

As before, we will use the same symbols for random variables and their realisations.

Lemma 4.3. *Let $((\sigma_\bullet, \sigma_\circ), U)$ be distributed according to $\Psi^{1/0}$, and let $\xi_\tau, \xi_{\tau\tau'}$ be as constructed above. Then the probability of a realisation $(\xi_\tau, \xi_{\tau\tau'}) \in \{0,1\}^{E_1} \times \{0,1\}^{E_1}$ is proportional to $\mathsf{mATRC}_K(\,\cdot\,|\, v_L \overset{\xi_\tau}{\longleftrightarrow} v_R)$.*

Proof. We build on [GP23, Proof of Lemma 7.1]. One has to examine which values of $(\sigma_\bullet, \sigma_\circ, U)$ give a fixed pair $\xi_\tau, \xi_{\tau\tau'} \in \{0,1\}^{E_1}$. Recall first that $\sigma_\bullet$ has to be constant on $\xi_{\tau\tau'}$ and that $\sigma_\circ$ has to be constant on ξ_τ^*. By (10), the probability of a quadruplet $(\sigma_\bullet, \sigma_\circ, \xi_\tau, \xi_{\tau\tau'})$ is

$$\begin{aligned}
&\mathsf{Spin}_\mathcal{D}^{+,+-}(\sigma_\bullet,\sigma_\circ)\, \mathbb{1}_{\sigma_\bullet\sim\xi_{\tau\tau'}}\, \mathbb{1}_{\sigma_\circ\sim\xi_\tau^*}\, \mathbb{1}_{\xi_\tau\subseteq\xi_{\tau\tau'}}\, \mathbb{1}_{\xi_{\tau\tau'}\setminus\xi_\tau\subseteq E} \\
&\quad\cdot \prod_{t\in T_{5,6}^\mathrm{i}} \big(\tfrac{1}{\mathbf{c}}\,(\mathbb{1}_{t\in\xi_\tau} + \mathbb{1}_{t\in\xi_{\tau\tau'}^*}) + \tfrac{\mathbf{c}-2}{\mathbf{c}}\,\mathbb{1}_{t\in\xi_{\tau\tau'}\setminus\xi_\tau}\big) \prod_{t\in T_{5,6}^\mathrm{b}} \big(\tfrac{1}{\mathbf{c}_\mathrm{b}}\,\mathbb{1}_{t\in\xi_\tau} + \tfrac{\mathbf{c}_\mathrm{b}-1}{\mathbf{c}_\mathrm{b}}\,\mathbb{1}_{t\in\xi_{\tau\tau'}^*}\big) \\
&\propto \mathbb{1}_{\Sigma_\Lambda^+}(\sigma_\bullet)\mathbb{1}_{\Sigma_{\Lambda'}^{+-}}(\sigma_\circ)\, \mathbb{1}_{\sigma_\bullet\sim\xi_{\tau\tau'}}\, \mathbb{1}_{\sigma_\circ\sim\xi_\tau^*}\, \mathbb{1}_{\xi_\tau\subseteq\xi_{\tau\tau'}}\, \mathbb{1}_{\xi_{\tau\tau'}\setminus\xi_\tau\subseteq E} \\
&\quad\cdot \prod_{t\in T_{5,6}^\mathrm{i}} \big(\mathbb{1}_{t\in\xi_\tau} + \mathbb{1}_{t\in\xi_{\tau\tau'}^*} + (\mathbf{c}-2)\mathbb{1}_{t\in\xi_{\tau\tau'}\setminus\xi_\tau}\big) \prod_{t\in T_{5,6}^\mathrm{b}} \big(\mathbb{1}_{t\in\xi_\tau} + (\mathbf{c}_\mathrm{b}-1)\mathbb{1}_{t\in\xi_{\tau\tau'}^*}\big), \qquad (13)
\end{aligned}$$

where we used the shorthand $T^{\#}_{5,6} \equiv T^{\#}_{5,6}(\sigma_\bullet, \sigma_\circ)$ and the fact that

$$\mathbb{1}_{\mathrm{ice}}(\sigma_\bullet, \sigma_\circ)\, \mathbb{1}_{\sigma_\bullet \sim \xi_{\tau\tau'}}\, \mathbb{1}_{\sigma_\circ \sim \xi^*_\tau}\, \mathbb{1}_{\xi_\tau \subseteq \xi_{\tau\tau'}} = \mathbb{1}_{\sigma_\bullet \sim \xi_{\tau\tau'}}\, \mathbb{1}_{\sigma_\circ \sim \xi^*_\tau}\, \mathbb{1}_{\xi_\tau \subseteq \xi_{\tau\tau'}}.$$

Observe then that under the event $\{\sigma_\bullet \sim \xi_{\tau\tau'},\, \sigma_\circ \sim \xi^*_\tau,\, \xi_\tau \subseteq \xi_{\tau\tau'},\, \xi_{\tau\tau'} \setminus \xi_\tau \subseteq E\}$,

$$\mathbb{1}_{T^{\mathrm{i}}_{5,6}}(t) = \mathbb{1}_{\xi_\tau}(t)\mathbb{1}_{\sigma_\circ \sim e^*_t} + \mathbb{1}_{\xi^*_{\tau\tau'}}(t)\mathbb{1}_{\sigma_\bullet \sim e_t} + \mathbb{1}_{\xi_{\tau\tau'}\setminus\xi_\tau}(t),$$
$$\mathbb{1}_{T^{\mathrm{b}}_{5,6}}(t) = \mathbb{1}_{\xi_\tau}(t)\mathbb{1}_{\sigma_\circ \sim e^*_t} + \mathbb{1}_{\xi^*_{\tau\tau'}}(t)\mathbb{1}_{\sigma_\bullet \sim e_t}.$$

Using this, (13) becomes

$$\begin{aligned}
&\mathbb{1}_{\Sigma^+_\Lambda}(\sigma_\bullet)\mathbb{1}_{\Sigma^{+-}_{\Lambda'}}(\sigma_\circ)\mathbb{1}_{\sigma_\bullet\sim\xi_{\tau\tau'}}\mathbb{1}_{\sigma_\circ\sim\xi^*_\tau}\mathbb{1}_{\xi_\tau\subseteq\xi_{\tau\tau'}}\mathbb{1}_{\xi_{\tau\tau'}\setminus\xi_\tau\subseteq E}\\
&\cdot\prod_{t\in A^{\mathrm{i}}}\big(\mathbb{1}_{t\in\xi_\tau}+\mathbb{1}_{t\in\xi^*_{\tau\tau'}}+(\mathbf{c}-2)\mathbb{1}_{t\in\xi_{\tau\tau'}\setminus\xi_\tau}\big)^{\mathbb{1}_{T^{\mathrm{i}}_{5,6}}(t)}\prod_{t\in\partial A}\big(\mathbb{1}_{t\in\xi_\tau}+(\mathbf{c}_{\mathrm{b}}-1)\mathbb{1}_{t\in\xi^*_{\tau\tau'}}\big)^{\mathbb{1}_{T^{\mathrm{b}}_{5,6}}(t)}\\
=\;&\mathbb{1}_{\Sigma^+_\Lambda}(\sigma_\bullet)\mathbb{1}_{\Sigma^{+-}_{\Lambda'}}(\sigma_\circ)\mathbb{1}_{\sigma_\bullet\sim\xi_{\tau\tau'}}\mathbb{1}_{\sigma_\circ\sim\xi^*_\tau}\mathbb{1}_{\xi_\tau\subseteq\xi_{\tau\tau'}}\mathbb{1}_{\xi_{\tau\tau'}\setminus\xi_\tau\subseteq E}(\mathbf{c}-2)^{|\xi_{\tau\tau'}\setminus\xi_\tau|}\prod_{e\in E_{\mathrm{b}}\setminus\xi_{\tau\tau'}}(\mathbf{c}_{\mathrm{b}}-1)^{\mathbb{1}_{\sigma_\bullet\sim e}}.
\end{aligned}$$

Observe that $\sigma_\circ \in \Sigma^{+-}_{\Lambda'}$ and $\sigma_\circ \sim \xi^*_\tau$ imply that v_L and v_R are connected in ξ_τ (see Figure 7). To obtain the probability of a pair $(\xi_\tau, \xi_{\tau\tau'})$ satisfying $v_L \overset{\xi_\tau}{\longleftrightarrow} v_R$ (up to multiplicative constant), we need to sum the last expression over $(\sigma_\bullet, \sigma_\circ)$.

The configurations $\sigma^\circ \in \Sigma^{+-}_{\Lambda'}$ with $\sigma_\circ \sim \xi^*_\tau$ are in bijective correspondence with assignments of ±1 to the clusters of ξ^*_τ in $K'_\sim$ that are contained in Λ', whence there exist $2^{\kappa_{K'_\sim}(\xi^*_\tau)-1}$ of them. By Euler's formula,

$$\kappa_{K'_\sim}(\xi^*_\tau) - 1 = \kappa_K(\xi_\tau) + |\xi_\tau| - |\mathbb{V}_{E_1}|.$$

In particular, the probability of a triplet $(\xi_\tau, \xi_{\tau\tau'}, \sigma_\bullet)$ with $v_L \overset{\xi_\tau}{\longleftrightarrow} v_R$ is proportional to

$$\mathbb{1}_{\Sigma^+_\Lambda}(\sigma_\bullet)\mathbb{1}_{\sigma_\bullet\sim\xi_{\tau\tau'}}\mathbb{1}_{\xi_\tau\subseteq\xi_{\tau\tau'}}\mathbb{1}_{\xi_{\tau\tau'}\setminus\xi_\tau\subseteq E}2^{\kappa_K(\xi_\tau)}2^{|\xi_\tau|}(\mathbf{c}-2)^{|\xi_{\tau\tau'}\setminus\xi_\tau|}\prod_{e\in E_{\mathrm{b}}\setminus\xi_{\tau\tau'}}(\mathbf{c}_{\mathrm{b}}-1)^{\mathbb{1}_{\sigma_\bullet\sim e}}. \quad (14)$$

We now sum this expression over $\sigma_\bullet$. As in the $\sigma_\circ$ case, this is equivalent to summing over assignments of ±1 values to the inner clusters of $\xi_{\tau\tau'}$ and a fixed $+1$ spin to the boundary clusters. Denote the sets of the inner clusters by $\mathrm{cl}^{\mathrm{i}}_\Lambda(\xi_{\tau\tau'})$ and the set of the boundary clusters by $\mathrm{cl}^{\mathrm{b}}_\Lambda(\xi_{\tau\tau'})$. Writing cl, cl^{i} and cl^{b} for brevity, we get:

- for any edge $e \in E^+_{\mathrm{b}}$, every $\sigma_\bullet \in \Sigma^+_\Lambda$ is constant $+1$ on the endpoints of e, and hence $\sigma_\bullet \sim e$;
- for any $\sigma_\bullet \in \Sigma^+_\Lambda$ with $\sigma_\bullet \sim \xi_{\tau\tau'}$, each edge $e \in E^-_{\mathrm{b}} \setminus \xi_{\tau\tau'}$ satisfies $\sigma_\bullet \sim e$ in precisely two cases: (i) if $e \in \partial^{\mathrm{edge}}\mathcal{C}$ for a boundary cluster $\mathcal{C} \in \mathrm{cl}^{\mathrm{b}}$; (ii) if $e \in \partial^{\mathrm{edge}}\mathcal{C}$ for an inner cluster $\mathcal{C} \in \mathrm{cl}^{\mathrm{i}}$ (which is then unique and) on which $\sigma_\bullet = +1$.

Therefore,

$$\begin{aligned}
\sum_{\substack{\sigma_\bullet\in\Sigma^+_\Lambda\\ \sigma_\bullet\sim\xi_{\tau\tau'}}}\prod_{e\in E_{\mathrm{b}}\setminus\xi_{\tau\tau'}}(\mathbf{c}_{\mathrm{b}}-1)^{\mathbb{1}_{\sigma_\bullet\sim e}} &= (\mathbf{c}_{\mathrm{b}}-1)^{|E^+_{\mathrm{b}}\setminus\xi_{\tau\tau'}|}\sum_{\substack{\sigma_\bullet\in\Sigma^+_\Lambda\\ \sigma_\bullet\sim\xi_{\tau\tau'}}}\prod_{e\in E^-_{\mathrm{b}}\setminus\xi_{\tau\tau'}}(\mathbf{c}_{\mathrm{b}}-1)^{\mathbb{1}_{\sigma_\bullet\sim e}}\\
&= (\mathbf{c}_{\mathrm{b}}-1)^{|E^+_{\mathrm{b}}\setminus\xi_{\tau\tau'}|}\Bigg(\prod_{\mathcal{C}\in\mathrm{cl}^{\mathrm{b}}}(\mathbf{c}_{\mathrm{b}}-1)^{|E^-_{\mathrm{b}}\cap\partial^{\mathrm{edge}}\mathcal{C}|}\Bigg)\Bigg(\prod_{\mathcal{C}\in\mathrm{cl}^{\mathrm{i}}}\big(1+(\mathbf{c}_{\mathrm{b}}-1)^{|E^-_{\mathrm{b}}\cap\partial^{\mathrm{edge}}\mathcal{C}|}\big)\Bigg).
\end{aligned}$$

Substituting the last display in the sum of (14) over $\sigma_\bullet$, we get mATRC_K as required. □

4.4. **Properties of the modified ATRC.** In this section, we will derive basic properties of the modified ATRC measure that will be instrumental in its analysis in Section 8 and in the proof of Theorem 2. We continue in the setting of the previous section.

Computation of the marginals on $E \cup E_{\mathrm{b}}^+$. Recall the Defintion 1 of the modified ATRC measure and the coupling from Lemma 4.3. To study the cluster of v_L, v_R in ξ_τ, it suffices to consider the restriction of ξ_τ to $E \cup E_{\mathrm{b}}^+$. We first introduce the corresponding measure.

Definition 2. Let $K^+ \equiv K^+_{n,m}$ be the graph $(\mathbb{V}_{E\cup E_{\mathrm{b}}^+}, E \cup E_{\mathrm{b}}^+)$. We regard $a \in \{0,1\}^{E\cup E_{\mathrm{b}}^+}$ as a spanning sub-graph of K^+. Define $\kappa_{K^+}(a)$ as the number of clusters in a, and let $\mathrm{cl}_\Lambda(a)$ be the set of clusters in a that intersect Λ. Moreover, given a subset $C \subseteq \mathbb{V}_{E\cup E_{\mathrm{b}}^+}$, define its *lower boundary index* by $I(C) = \sum_{e\in E_{\mathrm{b}}^-} |e \cap C|$. Define the modified ATRC probability measure $\mathsf{mATRC}^+_{n,m} \equiv \mathsf{mATRC}_{K^+}$ on $\{0,1\}^{E\cup E_{\mathrm{b}}^+} \times \{0,1\}^{E\cup E_{\mathrm{b}}^+}$ by

$$\mathsf{mATRC}_{K^+}(a,b) \propto \mathbb{1}_{a\subseteq b}\mathbb{1}_{b\setminus a\subseteq E}2^{|a\cap E|}\big(\tfrac{2}{\mathbf{c}_{\mathrm{b}}-1}\big)^{|a\cap E_{\mathrm{b}}^+|}(\mathbf{c}-2)^{|b\setminus a|} \cdot 2^{\kappa_{K^+}(a)} \prod_{C\in\mathrm{cl}_\Lambda(b)} \big(\mathbb{1}_{C\subseteq\Lambda} + \mathbf{c}_{\mathrm{b}}^{I(C)}\big).$$

Lemma 4.4. *Let $(\xi_\tau, \xi_{\tau\tau'}) \sim \mathsf{mATRC}_K$. The law of $(\xi_\tau, \xi_{\tau\tau'})$ restricted to $E \cup E_{\mathrm{b}}^+$ is given by mATRC_{K^+}.*

Proof. Recall the definitions of $V = V_{n,m}$ and $\Lambda = \Lambda_{n,m}$ given is Section 4.3. Introduce the probability measure $\mathcal{Q}$ on quadruplets $(a,b,s,t) \in (\{0,1\}^{E_1})^2 \times (\{\pm1\}^V)^2$ by

$$\mathcal{Q}(a,b,s,t) \propto \mathbb{1}_{\Sigma_\Lambda^+}(t)\mathbb{1}_{a\subseteq b}\mathbb{1}_{b\setminus a\subset E}\mathbb{1}_{s\sim a}\mathbb{1}_{t\sim b}\prod_{e\in E_1}2^{a_e}(\mathbf{c}-2)^{b_e-a_e}\prod_{e\in E_{\mathrm{b}}}(\mathbf{c}_{\mathrm{b}}-1)^{\mathbb{1}_{t\sim e}(1-a_e)}, \tag{15}$$

where $t \in \Sigma_\Lambda^+$ if and only if $t(i) = 1$ for all $i \in V \setminus \Lambda$. The marginal of $\mathcal{Q}$ on a, b is mATRC_K. Indeed, summing (15) over s and using that $a = b$ on E_{b} gives (14) with $t = \sigma_\bullet$. Summing then over t and reasoning precisely as below (14), one obtains $\mathsf{mATRC}_K(a,b)$ as defined in Definition 1.

We will now compute the desired marginals of mATRC_K by summing (15). For a, s as above and $e \in E_1$, we write $s_e \sim a_e$ if $a_e = 0$ or $s \sim e$. We use the same notation for b, t as above. Observe that any $t \in \Sigma_\Lambda^+$ and $e \in E_{\mathrm{b}}^+$ satisfy $t \sim e$. We start by summing (15) over the values of a_e, b_e for $e \in E_{\mathrm{b}}^-$, and the values of s_i for $i \in \mathbb{V}_{E_{\mathrm{b}}^-} \setminus \Lambda$ (denoted $\sum^*_{a,b,s}$), which yields

$$\mathbb{1}_{\Sigma_\Lambda^+}(t)\prod_{e\in E}\mathbb{1}_{a_e\le b_e}\mathbb{1}_{s_e\sim a_e}\mathbb{1}_{t_e\sim b_e}2^{a_e}(\mathbf{c}-2)^{b_e-a_e}\prod_{e\in E_{\mathrm{b}}^+}2^{a_e}\mathbb{1}_{a_e=b_e}\mathbb{1}_{s_e\sim a_e}\mathbb{1}_{t_e\sim b_e}(\mathbf{c}_{\mathrm{b}}-1)^{1-a_e} \cdot \sum_{a,b,s}^{*}\prod_{e\in E_{\mathrm{b}}^-}2^{a_e}\mathbb{1}_{a_e=b_e}\mathbb{1}_{s_e\sim a_e}\mathbb{1}_{t_e\sim b_e}(\mathbf{c}_{\mathrm{b}}-1)^{\mathbb{1}_{t\sim e}(1-a_e)}. \tag{16}$$

Now, the sum in the second row above satisfies

$$\sum_{a,b,s}^{*}\prod_{e\in E_{\mathrm{b}}^{-}} 2^{a_e}\mathbb{1}_{a_e=b_e}\mathbb{1}_{s_e\sim a_e}\mathbb{1}_{t_e\sim b_e}(\mathbf{c}_{\mathrm{b}}-1)^{\mathbb{1}_{t\sim e}(1-a_e)} = \sum_{s}^{*}\prod_{e\in E_{\mathrm{b}}^{-}}\left(2\mathbb{1}_{s\sim e}\mathbb{1}_{t\sim e}+(\mathbf{c}_{\mathrm{b}}-1)^{\mathbb{1}_{t\sim e}}\right)$$
$$=\prod_{e\in E_{\mathrm{b}}^{-}}\left(2\mathbb{1}_{t\sim e}+2(\mathbf{c}_{\mathrm{b}}-1)^{\mathbb{1}_{t\sim e}}\right)=2^{|E_{\mathrm{b}}^{-}|}\prod_{e\in E_{\mathrm{b}}^{-}}\mathbf{c}_{\mathrm{b}}^{\mathbb{1}_{t\sim e}}. \tag{17}$$

We then substitute (17) into (16) and sum the resulting expression over the remaining values of s and over $t\in\Sigma_\Lambda^+$, which for any $a,b\in\{0,1\}^{E\cup E_b^+}$ gives

$$\mathcal{Q}(a,b)\propto \mathbb{1}_{a\subseteq b}\prod_{e\in E}2^{a_e}(\mathbf{c}-2)^{b_e-a_e}\prod_{e\in E_{\mathrm{b}}^{+}}2^{a_e}\mathbb{1}_{a_e=b_e}(\mathbf{c}_{\mathrm{b}}-1)^{(1-a_e)}\sum_{s,t}\mathbb{1}_{s\sim a}\mathbb{1}_{t\sim b}\prod_{e\in E_{\mathrm{b}}^{-}}\mathbf{c}_{\mathrm{b}}^{t\sim e}.$$

Along the lines below (14) in the proof of Lemma 4.3, we see that

$$\sum_{s}\mathbb{1}_{s\sim a}=2^{\kappa_{K^+}(a)}\qquad\text{and}\qquad\sum_{t}\mathbb{1}_{t\sim b}\prod_{e\in E_{\mathrm{b}}^{-}}\mathbf{c}_{\mathrm{b}}^{t\sim e}=\prod_{C\in\mathrm{cl}_\Lambda(b)}\left(\mathbb{1}_{C\subseteq\Lambda}+\mathbf{c}_{\mathrm{b}}^{\sum_{e\in E_{\mathrm{b}}^{-}}|e\cap C|}\right).$$

□

Positive association. The motivation for considering the marginals of the modified ATRC measure on $E\cup E_{\mathrm{b}}^+$ is that they satisfy the strong FKG property (see Section 3). This will allow us to 'sandwich' the associated modified ATRC measure between unmodified ATRC measures, and from this to deduce the convergence to the unique infinite-volume ATRC *Gibbs measure*; see Section 5.

Lemma 4.5. *The measure* mATRC_{K^+} *satisfies the strong FKG property.*

Proof. Recall that $\mathbf{c}_{\mathrm{b}}=e^\lambda$ with $\lambda>0$. We verify the Holley criterion [Hol74]. To shorten notation, given $a,b\in\{0,1\}^{E\cup E_{\mathrm{b}}^+}$ with $a\subseteq b$ and $a=b$ on E_b^+, we write $\mathsf{mATRC}_{K^+}(a_e,b_e\,|\,a_{e^c},b_{e^c})$ for the conditional probability

$$\mathsf{mATRC}_{K^+}\big((\xi_\tau(e),\xi_{\tau\tau'}(e))=(a_e,b_e)\,\big|\,(\xi_\tau,\xi_{\tau\tau'})=(a,b)\text{ on }(E\cup E_{\mathrm{b}}^+)\setminus\{e\}\big).$$

For $e=ij\in E$, denote by C_i^a,C_j^a the clusters of i,j in a_{e^c}, and by $\widetilde{C}_i^b,\widetilde{C}_j^b$ the clusters of i,j in $b_{e^c}\cup E_{\mathrm{b}}^+$. Observe that $\widetilde{C}_i^b\neq\widetilde{C}_j^b$ implies that $\widetilde{C}_i^b\subseteq\Lambda$ or $\widetilde{C}_j^b\subseteq\Lambda$. Then, for any $a,b\in\{0,1\}^{E\cup E_{\mathrm{b}}^+}$ with $a\subseteq b$ and $a=b$ on E_b^+,

$$\mathsf{mATRC}_{K^+}(a_e,b_e\,|\,a_{e^c},b_{e^c})=\left(2^{a_e}\tfrac{\mathbf{c}}{\mathbf{c}+1}\right)^{\mathbb{1}_{C_i^a=C_j^a}}(\mathbf{c}-2)^{b_e-a_e}\tfrac{f(b)^{b_e}}{1+(\mathbf{c}-1)f(b)}, \tag{18}$$

where

$$f(b)=\mathbb{1}_{\widetilde{C}_i^b=\widetilde{C}_j^b}+\alpha\left(\mathbb{1}_{\widetilde{C}_i^b\neq\widetilde{C}_j^b\not\subseteq\Lambda}+\mathbb{1}_{\Lambda\not\supseteq\widetilde{C}_i^b\neq\widetilde{C}_j^b}\right)+\beta\mathbb{1}_{\Lambda\supseteq\widetilde{C}_i^b\neq\widetilde{C}_j^b\subseteq\Lambda}$$

and

$$\alpha=\frac{e^{\lambda I(\widetilde{C}_i^b)}}{1+e^{\lambda I(\widetilde{C}_i^b)}},\qquad \beta=\frac{1+e^{\lambda I(\widetilde{C}_i^b)+\lambda I(\widetilde{C}_j^b)}}{(1+e^{\lambda I(\widetilde{C}_i^b)})(1+e^{\lambda I(\widetilde{C}_j^b)})}.$$

Note first that $\mathbf{c}\geq 1$ (in fact $\mathbf{c}>2$) and $\mathbb{1}_{C_i^a=C_j^a}$ is increasing in a. Moreover, $\beta\leq\alpha\leq 1$ and both α and β are increasing in $I(\widetilde{C}_i^b)$ and $I(\widetilde{C}_j^b)$ and hence in b. Therefore $f\geq 1$ is increasing in b. This implies that $\mathsf{mATRC}_{K^+}(1,1\,|\,a_{e^c},b_{e^c})$ is increasing in a,b and $\mathsf{mATRC}_{K^+}(0,0\,|\,a_{e^c},b_{e^c})$ is decreasing in a,b.

For $e=\{i,j\}\in E_b^+$, one has for any $a_e=b_e$,

$$\mathsf{mATRC}_{K^+}(a_e,b_e\,|\,a_{e^c},b_{e^c})=\left(2^{a_e}\tfrac{\mathbf{c}_{\mathrm{b}}}{\mathbf{c}_{\mathrm{b}}+1}\right)^{\mathbb{1}_{C_i^a=C_j^a}}\left(\tfrac{f(b)}{\mathbf{c}_{\mathrm{b}}-1}\right)^{a_e}\tfrac{\mathbf{c}_{\mathrm{b}}-1}{\mathbf{c}_{\mathrm{b}}-1+f(b)}, \tag{19}$$

where $f(b)$ is as above. Recall that $\mathbf{c}_{\mathrm{b}} = e^{\lambda} > 1$ and that $\mathbb{1}_{C_i^a = C_j^a}$ and f are increasing in a and b, respectively. This implies that $\mathsf{mATRC}_{K^+}(1, 1 \,|\, a_{e^c}, b_{e^c})$ is increasing in a, b, and hence $\mathsf{mATRC}_{K^+}(0, 0 \,|\, a_{e^c}, b_{e^c})$ is decreasing in a, b, and the proof is complete. □

Finite energy. Another feature of the marginals of the modifed ATRC measure is that they satisfy the finite-energy property.

Lemma 4.6. *The exists a constant $c > 0$ such that, for any $e \in E$ and any $a, b \in \{0,1\}^{E \cup E_{\mathrm{b}}^+}$ with $a \subseteq b$ and $b \setminus a \subseteq E$,*

$$\mathsf{mATRC}_{K^+}\big((\xi_\tau(e), \xi_{\tau\tau'}(e)) = (a_e, b_e) \,\big|\, (\xi_\tau, \xi_{\tau\tau'}) = (a, b) \text{ on } (E \cup E_{\mathrm{b}}^+) \setminus \{e\}\big) > c.$$

The statement also holds for $e \in E_{\mathrm{b}}^+$, provided that $a_e = b_e$.

The above lemma follows directly from the explicit expressions (18)-(19) of the conditional probabilities computed in the proof of Lemma 4.5.

Controlling regions properties. The following *controlling regions* property was introduced in [Ale04].

Lemma 4.7. *Let $(\xi_\tau, \xi_{\tau\tau'})$ be distributed according to mATRC_K. Let $F_1 \subset F_2 \subset E$, and let $\tilde{\xi}_\tau, \tilde{\xi}_{\tau\tau'} \in \{0,1\}^{F_2 \setminus F_1}$ be such that there exists circuits in $\tilde{\xi}_\tau^*$ and in $\tilde{\xi}_{\tau\tau'}$ that both surround F_1. Then, conditionally on $(\xi_\tau|_{F_2 \setminus F_1}, \xi_{\tau\tau'}|_{F_2 \setminus F_1}) = (\tilde{\xi}_\tau, \tilde{\xi}_{\tau\tau'})$, the values of $(\xi_\tau|_{F_1}, \xi_{\tau\tau'}|_{F_1})$ and $(\xi_\tau|_{E \setminus F_2}, \xi_{\tau\tau'}|_{E \setminus F_2})$ are independent.*

The proof is very similar to that of Lemma 4.8 below, though notationally heavier, whence we omit it here.

4.5. **Interface and conditional measure.** We continue in the setting of the previous section. Recall the coupling $\Psi^{1/0}$ of the six-vertex spin random variable $(\sigma_\bullet, \sigma_\circ) \sim \mathsf{Spin}_{\mathcal{D}}^{+,+-}$ and the modified ATRC pair $(\xi_\tau, \xi_{\tau\tau'}) \sim \mathsf{mATRC}_K$. In view of the more advantageous properties of mATRC_{K^+}, we will be investigating this measure rather than mATRC_K, whence we formulate the following definitions for the restrictions of $(\xi_\tau, \xi_{\tau\tau'})$ to $E \cup E_{\mathrm{b}}^+$.

Interface. Since $\sigma_\circ$ must be constant on edges in ξ_τ^* while being subject to Dobrushin boundary conditions ($\sigma_\circ \in \Sigma_{\Lambda'}^{+-}$), one automatically has that v_L and v_R are connected in ξ_τ. Let $\mathcal{C}_{v_L, v_R} \subseteq E \cup E_{\mathrm{b}}^+ \subset \mathbb{E}^\bullet$ be the set of edges that are connected to v_L, v_R in ξ_τ restricted to $E \cup E_{\mathrm{b}}^+$. Observe that there exists a unique closed curve $\gamma \subset \mathbb{R}^2$ given by the union of line-segments between endpoints of edges in $*\partial_{\mathbb{L}_\bullet}^{\mathrm{ex}} \mathcal{C}_{v_L, v_R}$ such that the edges in $\mathcal{C}_{v_L, v_R}$ are contained in the bounded connected component of $\mathbb{R}^2 \setminus \gamma$. Define the *surrounding polygon* $\mathcal{P}$ as the closure of the bounded connected component of $\mathbb{R}^2 \setminus \gamma$ (so that $\gamma \subset \mathcal{P}$). See Fig. 9. We now define our notion of interface:

- $\Gamma = \Gamma_{\mathsf{ATRC}} := \mathcal{P} \cap \mathbb{L}_\bullet \subseteq V$, which is simply connected,
- $\mathcal{A}_1 \subseteq V$ is the set of vertices above Γ, that is, the vertices in $V \setminus \Gamma$ that are connected to $\partial^{\mathrm{in}} V \cap \mathbb{H}^+$ in the graph induced by $V \setminus \Gamma$,
- $\mathcal{A}_0 \subseteq V$ is the set of vertices below Γ, that is, the vertices in $V \setminus \Gamma$ that are connected to $\partial^{\mathrm{in}} V \cap \mathbb{H}^-$ in the graph induced by $V \setminus \Gamma$.

Moreover, let $\Gamma' := \mathcal{P} \cap \mathbb{L}_\circ$.

Expression of the conditional measure. We now investigate the measure mATRC_{K^+} conditionally on a realisation of $\mathcal{C}_{v_L, v_R}$, which will be used in the study of the relaxation to pure phases away from the interface. Given a realisation C of $\mathcal{C}_{v_L, v_R}$, we write

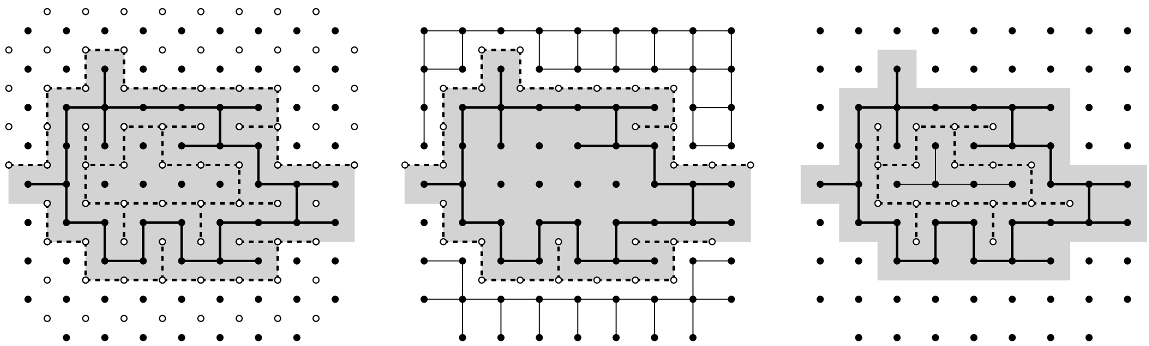

FIGURE 9. Left: A realisation C of $\mathcal{C}_{v_L,v_R}$ (solid edges) in K for $n = m = 3$, the dual $*\partial^{\mathrm{ex}}_{K^+}C$ of its external edge-boundary in K^+ (dashed edges), and the surrounding polygon $\mathcal{P}_C$ (grey). Center: The edge-set E^{ex}_C (thin solid edges). Right: The edge-set E^{in}_C (thin solid edges).

$\mathcal{P}_C, \Gamma_C, \Gamma'_C$ for the corresponding realisations of $\mathcal{P}, \Gamma, \Gamma'$, respectively, and we define

$$E^{\partial\mathrm{ex}}_C := \{e \in \partial^{\mathrm{ex}}_{K^+}C : e \cup e^* \not\subset \mathcal{P}_C\}, \qquad E^{\mathrm{ex}}_C := E^{\partial\mathrm{ex}}_C \cup \{e \in E \cup E^+_{\mathrm{b}} : e \subset \mathbb{R}^2 \setminus \mathcal{P}_C\}.$$

Let $K_C := (\mathbb{V}_{E^{\mathrm{ex}}_C \setminus E^{\partial\mathrm{ex}}_C}, E^{\mathrm{ex}}_C \setminus E^{\partial\mathrm{ex}}_C)$, and let K^1_C be the graph obtained from $(\mathbb{V}_{E^{\mathrm{ex}}_C}, E^{\mathrm{ex}}_C)$ by identifying the vertices connected by C. See Fig. 9. First, we will derive an explicit expression for the conditional measure.

Lemma 4.8. *For a realisation C of $\mathcal{C}_{v_L,v_R}$ and $a^{\mathrm{ex}}, b^{\mathrm{ex}} \in \{0,1\}^{E^{\mathrm{ex}}_C}$ with $a^{\mathrm{ex}} \cap E^{\partial\mathrm{ex}}_C = \varnothing$,*

$$\mathsf{mATRC}_{K^+}\big((\xi_\tau|_{E^{\mathrm{ex}}_C}, \xi_{\tau\tau'}|_{E^{\mathrm{ex}}_C}) = (a^{\mathrm{ex}}, b^{\mathrm{ex}}) \,\big|\, \mathcal{C}_{v_L,v_R} = C\big) \propto \mathbb{1}_{a^{\mathrm{ex}} \subseteq b^{\mathrm{ex}}}\, \mathbb{1}_{b^{\mathrm{ex}} \setminus a^{\mathrm{ex}} \subseteq E}$$
$$\cdot\, 2^{|a^{\mathrm{ex}} \cap E|} \left(\tfrac{2}{\mathbf{c}_{\mathrm{b}} - 1}\right)^{|a^{\mathrm{ex}} \cap E^+_{\mathrm{b}}|} (\mathbf{c} - 2)^{|b^{\mathrm{ex}} \setminus a^{\mathrm{ex}}|}\, 2^{\kappa_{K_C}(a^{\mathrm{ex}})} \prod_{\mathcal{C} \in \mathrm{cl}_{K^1_C}(b^{\mathrm{ex}})} \big(\mathbb{1}_{\mathcal{C} \subseteq \Lambda} + \mathbf{c}_{\mathrm{b}}^{I(\mathcal{C})}\big).$$

where $\kappa_{K_C}(a^{\mathrm{ex}})$ is the number of clusters in the spanning sub-graph of K_C with edge-set a^{ex}, and $\mathrm{cl}_{K^1_C}(b^{\mathrm{ex}})$ is the set of clusters in the spanning sub-graph of K^1_C with edge-set b^{ex} that intersect Λ.

Proof. Define $E^{\partial\mathrm{in}}_C := \partial^{\mathrm{ex}}C \setminus E^{\partial\mathrm{ex}}_C$ and $E^{\mathrm{in}}_C := (E \cup E^+_{\mathrm{b}}) \setminus (E^{\mathrm{ex}}_C \cup C)$, so that $E \cup E^+_{\mathrm{b}}$ is the disjoint union of E^{in}_C, E^{ex}_C and C, see Fig. 9. Let $G_C := (\mathbb{V}_{E^{\mathrm{in}}_C \setminus E^{\partial\mathrm{in}}_C}, E^{\mathrm{in}}_C \setminus E^{\partial\mathrm{in}}_C)$, and let G^1_C be the graph obtained from $(\mathbb{V}_{E^{\mathrm{in}}_C}, E^{\mathrm{in}}_C)$ by identifying the vertices connected by C. Since $\xi_\tau \subseteq \xi_{\tau\tau'}$ almost surely, we clearly have

$$\mathsf{mATRC}_{K^+}\big((\xi_\tau|_{E^{\mathrm{ex}}_C}, \xi_{\tau\tau'}|_{E^{\mathrm{ex}}_C}) = (a^{\mathrm{ex}}, b^{\mathrm{ex}}) \,\big|\, \mathcal{C}_{v_L,v_R} = C\big)$$
$$\propto \sum_{a^{\mathrm{in}}, b^{\mathrm{in}}} \mathsf{mATRC}_{K^+}\big((\xi_\tau, \xi_{\tau\tau'}) = (a^{\mathrm{ex}} \cup a^{\mathrm{in}} \cup C, b^{\mathrm{ex}} \cup b^{\mathrm{in}} \cup C)\big),$$

where the sum is over all $a^{\mathrm{in}}, b^{\mathrm{in}} \in \{0,1\}^{E^{\mathrm{in}}_C \cup E^{\partial\mathrm{in}}_C}$ with $a^{\mathrm{in}} \cap E^{\partial\mathrm{in}}_C = \varnothing$. Fix such a pair, and set $a = a^{\mathrm{ex}} \cup a^{\mathrm{in}} \cup C$ and $b = b^{\mathrm{ex}} \cup b^{\mathrm{in}} \cup C$. Since C is disjoint from and separates K_C and G_C, we clearly have

$$\kappa_K(a) = \kappa_{K_C}(a^{\mathrm{ex}}) + \kappa_{G_C}(a^{\mathrm{in}}) + 1.$$

Moreover, by the definitions of K^1_C and G^1_C (look again at Fig. 9), it holds that

$$\mathrm{cl}_\Lambda(b) = \mathrm{cl}_{K^1_C}(b^{\mathrm{ex}}) \sqcup \big(\mathrm{cl}_{G^1_C}(b^{\mathrm{in}}) \setminus \{\text{cluster of } C \text{ in } b^{\mathrm{in}}\}\big),$$

and every cluster $\mathcal{C}$ in the second set satisfies $\mathcal{C} \subseteq \Lambda$ and has lower boundary index $I(\mathcal{C}) = 0$, whence

$$\prod_{\mathcal{C}\in \mathrm{cl}_\Lambda(b)} \left(\mathbb{1}_{\mathcal{C}\subseteq\Lambda} + \mathbf{c}_{\mathrm{b}}^{I(\mathcal{C})}\right) = \prod_{\mathcal{C}\in \mathrm{cl}_{K_C^1}(b^{\mathrm{ex}})} \left(\mathbb{1}_{\mathcal{C}\subseteq\Lambda} + \mathbf{c}_{\mathrm{b}}^{I(\mathcal{C})}\right) \cdot 2^{\kappa_{G_C^1}(b^{\mathrm{in}})-1},$$

and the proof is complete. □

Proximity of interfaces. Consider the coupling measure $\Psi^{1/0}$ constructed in Section 4. A feature of this coupling, that is crucial for our purposes, is that the respective interfaces stay close to each other with high probability. To measure the distance between interfaces, we will work with the *one-sided Hausdorff distance* defined by

$$\mathrm{d_H}(R,S) := \sup_{x\in R}\inf_{y\in S} \mathrm{d}_\infty(x,y), \qquad R,S\subset\mathbb{R}^2.$$

Given a bi-infinite connected set $C \subset \mathbb{L}_\bullet \cap (\mathbb{R}\times[-c,c])$ with $C\cap(\mathbb{R}\times\{\pm c\}) \neq \varnothing$, we say that a subset $R\subset\mathbb{R}^2$ is (weakly) *above* C if it is contained in the closure of the connected component of $(0,c+1)$ in $\mathbb{R}^2\setminus C$, where we identify C with the union of line segments between the endpoints of the edges in $\mathbb{E}_C$. We say that R is *below* C if it is contained in the closure of the connected component of $(0,-c-1)$ in $\mathbb{R}^2\setminus C$. We make the analogous definitions for finite connected sets by extending them in a natural way to bi-finite connected sets, and for connected subsets $C\subset\mathbb{L}_\circ$. Recall the definition of the upper and lower envelopes $\Gamma_{\mathsf{FK}}^{\pm,n}$ in G_n defined in Section 1, and define $\Gamma_{\mathsf{FK}}^{\pm,n,m}$ in $G = G_{n,m}$ analogously.

Lemma 4.9. *There exist constants $C, c > 0$ such that, for any $k\geq 1$,*

$$\Psi^{1/0}(\mathrm{d_H}(\Gamma_{\mathsf{FK}}^{\pm,n,m},\Gamma_{\mathsf{ATRC}}) > k) \leq Cnmke^{-ck}.$$

Proof. We present the argument for $\Gamma_{\mathsf{FK}}^{+,n,m}$. The statement for $\Gamma_{\mathsf{FK}}^{-,n,m}$ is proved in an analogous fashion. Consider the coupling $\Psi^{1/0}$ of $\omega\sim\mathsf{FK}_G^{1/0}$, $(\sigma_\bullet,\sigma_\circ)\sim\mathsf{Spin}_{\mathcal{D}}^{+,+-}$ and $(\xi_\tau,\xi_{\tau\tau'})\sim\mathsf{mATRC}_K$ in Section 4. Let $\mathcal{C}' = \mathcal{C}'_{v'_L,v'_R}$ be the cluster of $v'_L := (-n-\frac12,-\frac12)$ and $v'_R := (n+\frac12,-\frac12)$ in ω^*, and observe that $\mathrm{d_H}(\Gamma_{\mathsf{FK}}^{+,n,m},\mathcal{C}')\leq\frac12$. Therefore, it suffices to bound the probability

$$\Psi^{1/0}(\mathrm{d_H}(\mathcal{C}',\Gamma_{\mathsf{ATRC}}) > k).$$

Let $p_-\subset\mathbb{V}_{*E}$ be the uppermost path connecting v'_L to v'_R such that $\sigma_\circ(u) = -1$ for all $u\in p_-$. Since $*\partial^{\mathrm{ex}}\mathcal{C}_{v_L,v_R}$ contains such a path, as well as a path p_+ that connects $(-n-\frac32,\frac12)$ to $(n+\frac32,\frac12)$ such that $\sigma_\circ(u) = +1$ for all $u\in p_+$, we have $\mathrm{d_H}(p_-,\Gamma_{\mathsf{ATRC}})\leq\frac12$. Therefore, it suffices to bound the probability

$$\Psi^{1/0}(\mathrm{d_H}(\mathcal{C}',p_-) > k).$$

In the coupling, it holds that $\sigma_\circ$ is constant -1 on $\mathcal{C}'$ (see Fig. 7). Thus, the path p_- cannot contain vertices below $\mathcal{C}'$. Hence, conditional on $\mathcal{C}' = C'$, the random variable $\mathrm{d_H}(\mathcal{C}',p_-)$ is measurable with respect to $\sigma_\circ$ restricted to the vertices in $\mathcal{D}\cap\mathbb{L}_\circ$ above C'. Let us determine the conditional law of $(\sigma_\bullet,\sigma_\circ)$ restricted to the vertices in $\mathcal{D}$ above C'.

Fix a realisation C' of $\mathcal{C}'$, and let $p_\gamma^\uparrow$ be the lower most path in ω above C'. Let $\mathcal{D}_{C'}$ be the subset of vertices in $\mathcal{D}$ above $p_\gamma^\uparrow$. Denote by $A_{C'}$ the set of tiles with at least one corner in $\mathcal{D}_{C'}$, and by $A_{C'}^{\mathrm{b}}$ the set of tiles with precisely one corner in $\mathcal{D}_{C'}$. Define

$$E_1^{C'} = \{e_t : t\in A_{C'}\},\quad E_{\mathrm{b}}^{C'} = \{e_t : t\in A_{C'}^{\mathrm{b}}\}\quad\text{and}\quad E^{C'} = E_1^{C'}\setminus E_{\mathrm{b}}^{C'}.$$

Set $G_{C'} = (\mathbb{V}_{E^{C'}}, E^{C'})$. Then, by Lemma 4.1 and the domain Markov property of FK-percolation, for any $\omega_0 \in \{0,1\}^{E^{C'}}$,

$$\Psi^{1/0}(\omega|_{E^{C'}} = \omega_0 \,|\, \mathcal{C}' = C') = \mathsf{FK}_G^{1/0}(\omega|_{E^{C'}} = \omega_0 \,|\, \mathcal{C}' = C') = \mathsf{FK}^1_{G_{C'}}(\omega|_{E^{C'}} = \omega_0).$$

Then, along the lines in the proof of Lemma 4.1,

$$\Psi^{1/0}\big((\sigma_\bullet|_{\mathcal{D}_{C'}\cap\mathbb{L}_\bullet}, \sigma_\circ|_{\mathcal{D}_{C'}\cap\mathbb{L}_\circ}) \in \cdot \,\big|\, \mathcal{C}' = C'\big) = \mathsf{Spin}^{+,+}_{\mathcal{D}_{C'}}|_{\mathcal{D}_{C'}},$$

where the latter is defined by setting, for $\sigma \in \{\pm1\}^{\mathbb{L}_\bullet} \times \{\pm1\}^{\mathbb{L}_\circ}$,

$$\mathsf{Spin}^{+,+}_{\mathcal{D}_{C'}}(\sigma) \propto \mathbf{c}^{|T^{\mathrm{i}}_{5,6}(\sigma)|}\, \mathbf{c}_{\mathrm{b}}^{|T^{\mathrm{b}}_{5,6}(\sigma)|}\, \mathbb{1}_{\sigma=+1 \text{ on } (\mathbb{L}_\bullet\cup\mathbb{L}_\circ)\setminus\mathcal{D}_{C'}}\, \mathbb{1}_{\mathrm{ice}}(\sigma),$$

where $T^{\mathrm{i}}_{5,6}(\sigma)$ and $T^{\mathrm{b}}_{5,6}(\sigma)$ are respectively the sets of tiles in $A_{C'} \setminus A^{\mathrm{b}}_{C'}$ and $A^{\mathrm{b}}_{C'}$ that are of types 5,6 in σ. Let us determine the conditional law of $(\xi_\tau, \xi_{\tau\tau'})$. Define $K_{C'} = (\mathbb{V}_{E_1^{C'}}, E_1^{C'})$, and let $K^1_{C'}$ be the graph obtained from $K_{C'}$ by identifying vertices in $\partial^{\mathrm{in}}_{\mathbb{L}_\bullet}\mathbb{V}_{E_1^{C'}}$ and those connected by $p^\uparrow_{C'}$. Observe that, conditional on $\mathcal{C}' = C'$, $\sigma_\bullet$ is constant $+1$ on the endpoints of edges in $E^{C'}_{\mathrm{b}}$. By a computation exactly analogous to that in the proof of Lemma 4.3,

$$\Psi^{1/0}\big((\xi_\tau|_{E_1^{C'}}, \xi_{\tau\tau'}|_{E_1^{C'}}) \in \cdot \,\big|\, \mathcal{C}' = C'\big) = \mathsf{mATRC}_{K_{C'}},$$

where $\mathsf{mATRC}_{K_{C'}}$ is defined by setting, for any $(a,b) \in \{0,1\}^{E_1^{C'}} \times \{0,1\}^{E_1^{C'}}$,

$$\mathsf{mATRC}_{K_{C'}}(a,b) \propto \mathbb{1}_{a\subseteq b}\mathbb{1}_{b\setminus a\subseteq E^{C'}} 2^{\kappa_{K_{C'}}(a)+\kappa_{K^1_{C'}}(b)} 2^{|a|}(\mathbf{c}-2)^{|b\setminus a|}(\mathbf{c}_{\mathrm{b}}-1)^{|E^{C'}_{\mathrm{b}}\setminus b|}.$$

It is easy to see that $\mathsf{mATRC}_{K_{C'}}$ is positively associated (a significant simplification of the proof of Lemma 4.5 applies). Together with the definition of $\mathsf{ATRC}^{1,1}_{G_{C'}}$, we deduce

$$\mathsf{mATRC}_{K_{C'}} \leq_{\mathrm{st}} \mathsf{mATRC}_{K_{C'}}(\cdot \,|\, \xi_\tau(e) = 1 \text{ for all } e \in E^{C'}_{\mathrm{b}}) = \mathsf{ATRC}^{1,1}_{G_{C'}}.$$

By uniform exponential decay of connection probabilities in $\mathsf{ATRC}^{1,1}_{G_{C'}}$ (Theorem 5) and since $E^{\mathrm{b}}_{C'}$ is a one-dimensional set, a result of [Ott25] and a conceptually simple but lengthy argument in [IOVW20] shows that connection probabilities in $\mathsf{mATRC}_{K_{C'}}$ decay exponentially as well.

Finally, since $\sigma_\circ \sim \xi^*_\tau$ and $\sigma_\circ = 1$ on $\mathbb{L}_\circ \setminus \mathcal{D}_{C'}$, any connected component of $\sigma_\circ = -1$ must be surrounded by a circuit in ξ_τ, we deduce

$$\Psi^{1/0}(\mathrm{d_H}(\mathcal{C}', p_-) > k \,|\, \mathcal{C}' = C') \leq \mathsf{mATRC}_{K_{C'}}(\exists\, i \in p^\uparrow_{C'} : i \overset{\xi_\tau}{\longleftrightarrow} i + \partial^{\mathrm{in}}\Lambda_k) \leq Cnmke^{-ck},$$

and the proof is complete. □

5. Weak mixing in the ATRC

The main results proved in this section are the single-edge relaxation of the ATRC, Proposition 3.2, and the ratio weak mixing property for the ATRC, Theorem 4.

In Subsections 5.1 and 5.2 we define the height function of the six-vertex model and use [GP23] to show that it relaxes exponentially to its infinite-volume limit. In Subsection 5.3, we show that this implies exponential relaxation for the ATRC measure with '0, 1' boundary conditions, stated in Proposition 5.9. In Subsection 5.4, we derive Proposition 3.2 from Proposition 5.9 and an imput from [ADG24], stated in Proposition 5.11. In Subsection 5.5, we show how to derive from Proposition 3.2 the *exponential* weak mixing property of the ATRC, stated in Theorem 7. Finally, we prove Theorem 4 using the classical work of Alexander [Ale98], Theorem 7 and the imput, Proposition 5.11, from [ADG24].

5.1. **The six-vertex height function.** The proof of Proposition 3.2 relies on exponential relaxation of the ATRC measures with '0, 1' boundary conditions. The latter will be established via the height function representation of the six-vertex model and the BKW coupling [BKW76] with FK-percolation, while it also requires some input from [GP23]. We first introduce the six-vertex height function and provide the necessary graph definitions.

5.1.1. *Six-vertex height function.* Recall the edge-orientations and spin representation of the six-vertex model introduced in Section 4.2. A six-vertex *height function* is an assignment of integers, called *heights*, to the vertices in both $\mathbb{L}_\bullet$ and $\mathbb{L}_\circ$. Given edge-orientations that satisfy the ice rule, define h on $\mathbb{L}_\bullet \cup \mathbb{L}_\circ$ as follows. Fix an integer height at some arbitrary fixed vertex. Then iteratively define the heights at other vertices by increasing the height by 1 when traversing an edge $e \in \mathbb{E}^\diamond$ from its left to its right (with respect to its assigned orientation) and decreasing the height by 1 when traversing an arrow from its right to its left; see Fig. 5. As with the spins, the procedure is self-consistent due to the ice rule. Note that the heights on $\mathbb{L}_\bullet$ and on $\mathbb{L}_\circ$ automatically have different parity. By convention, we set the parity on $\mathbb{L}_\bullet$ to be even. The gradient of h is in bijective correspondence with the edge-orientations and hence with the spin representation, up to a global spin flip, as demonstrated by the following relation (see Fig. 5):

$$h(u) - h(i) = \sigma_\bullet(i)\sigma_\circ(u) \qquad \text{for any } t \in \mathbb{L}_\diamond \text{ with } i \in e_t,\ u \in e_t^*.$$

On the other hand, up to a global spin flip, the spins are obtained from the height function by setting, for all $i \in \mathbb{L}_\bullet$ and $u \in \mathbb{L}_\circ$,

$$\sigma_\bullet(i) = \begin{cases} +1 & \text{if } h(i) = 0 \mod 4 \\ -1 & \text{if } h(i) = 2 \mod 4 \end{cases}, \quad \sigma_\circ(u) = \begin{cases} +1 & \text{if } h(u) = 1 \mod 4 \\ -1 & \text{if } h(u) = 3 \mod 4 \end{cases}. \tag{20}$$

This motivates the following definition.

Definition 3. A function $h : \mathbb{L}_\bullet \cup \mathbb{L}_\circ \to \mathbb{Z}$ is called a (six-vertex) height function if it satisfies the following:

- for any $t \in \mathbb{L}_\diamond$, $i \in t \cap \mathbb{L}_\bullet$, and $u \in t \cap \mathbb{L}_\circ$, one has $|h(i) - h(u)| = 1$,
- for any $i \in \mathbb{L}_\bullet$, one has $h(i) \in 2\mathbb{Z}$.

Denote the set of all height functions by Ω_{hf}. The type of a tile $t \in \mathbb{L}_\diamond$ in a height function h is given by the type of its gradient function; see Fig. 5. Fix $g \in \Omega_{\mathrm{hf}}$, parameters $\mathbf{c}, \mathbf{c}_{\mathrm{b}} > 0$ and a finite subset $\Delta \subset \mathbb{L}_\bullet \cup \mathbb{L}_\circ$. The set $A := A_\Delta \subset \mathbb{L}_\diamond$ of tiles of Δ is given by the tiles with at least one corner in Δ. The set $\partial A \subseteq A$ of boundary tiles of Δ is given by the tiles with precisely one corner in Δ. Let $\delta \subseteq \partial A$. The corresponding height function measure on $\mathbb{Z}^{\mathbb{L}_\bullet \cup \mathbb{L}_\circ}$ is defined by

$$\mathsf{HF}^g_{\Delta,\delta;\mathbf{c},\mathbf{c}_{\mathrm{b}}}(h) \propto \mathbf{c}^{|T^{A\setminus\delta}_{5,6}(h)|}\, \mathbf{c}_{\mathrm{b}}^{|T^{\delta}_{5,6}(h)|}\, \mathbb{1}_{\Omega_{\mathrm{hf}}}(h)\, \mathbb{1}_{h(x)=g(x)\,\forall x \in (\mathbb{L}_\bullet \cup \mathbb{L}_\circ)\setminus\Delta}\,, \tag{21}$$

where $T^{A\setminus\delta}_{5,6}(h)$ (resp. $T^{\delta}_{5,6}(h)$) is the set of tiles $t \in A \setminus \delta$ (resp. $t \in \delta$) of type 5-6 in h.

If g is constant n on $\mathbb{L}_\bullet$ and constant m on $\mathbb{L}_\circ$ (n even and $m = n \pm 1$), we simply write $\mathsf{HF}^{n,m}_{\Delta,\delta;\mathbf{c},\mathbf{c}_{\mathrm{b}}}$. When $\delta = \partial A$, we omit δ from the subscript. When $\delta = \varnothing$, we omit δ and $\mathbf{c}_{\mathrm{b}}$ from the subscript.

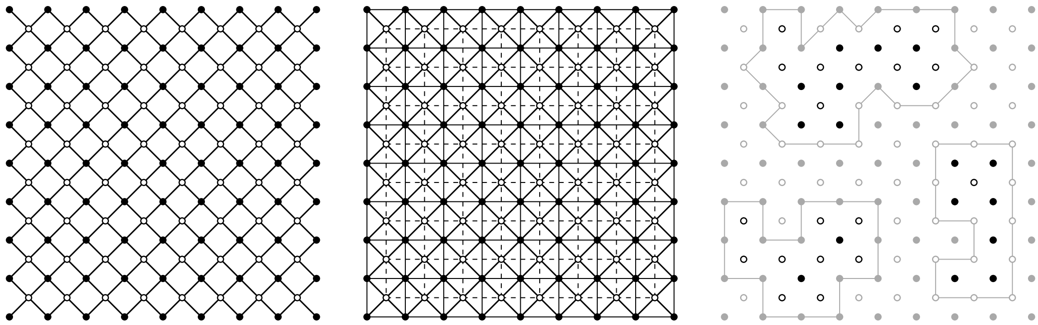

FIGURE 10. Left: part of the rotated square lattice $\mathbb{L}$. Its faces are the tiles in $\mathbb{L}_\diamond$. Center: part of the augmented lattice $\overline{\mathbb{L}}$. Right: $\mathbb{L}$-domains given by the vertices strictly within simple circuits in $\overline{\mathbb{L}}$, $\mathbb{L}_\bullet$, $\mathbb{L}_\circ$, respectively. The lower left and right $\mathbb{L}$-domains are even and odd, respectively.

5.1.2. *Rotated lattice and domains.* Consider the rotated square lattice $\mathbb{L}$ with vertex-set $\mathbb{L}_\bullet \cup \mathbb{L}_\circ$ and edges between nearest neighbours, that is, between vertices of Euclidean distance $1/\sqrt{2}$. Given $\Delta \subseteq \mathbb{L}$, we write $\Delta_\bullet = \Delta \cap \mathbb{L}_\bullet$ and $\Delta_\circ = \Delta \cap \mathbb{L}_\circ$. The augmented graph $\overline{\mathbb{L}}$ has the same vertex-set as $\mathbb{L}$ and all edges of $\mathbb{L}$, $\mathbb{L}_\bullet$ and $\mathbb{L}_\circ$, see Figure 10. We restrict the notion of simple circuits in $\overline{\mathbb{L}}$ to those that do not traverse both e and e^* for any $e \in \mathbb{E}^\bullet$, so that they can be embedded in $\mathbb{R}^2$. We identify such circuits with their planar embedding.

Definition 4 ($\mathbb{L}$-domains)**.** A finite subset $\mathcal{D} \subset \mathbb{L}$ is called an $\mathbb{L}$-domain if there exists a simple circuit C in $\overline{\mathbb{L}}$ such that $\mathcal{D}$ is given by the vertices of $\mathbb{L}$ strictly within C, that is, in the bounded connected component of $\mathbb{R}^2 \setminus C$. It is called *even* (respectively, *odd*) if $\partial^{\text{ex}}_{\mathbb{L}}\mathcal{D} \subset \mathbb{L}_\bullet$ (respectively, $\partial^{\text{ex}}_{\mathbb{L}}\mathcal{D} \subset \mathbb{L}_\circ$); see Figure 10.

Definition 5 ($\mathbb{L}_\bullet$- and $\mathbb{L}_\circ$-domains)**.** We identify circuits in $\mathbb{L}_\bullet$ and $\mathbb{L}_\circ$ with their sets of edges as well as with their planar embedding given by the union of line-segments between consecutive vertices. Given a simple circuit C in $\mathbb{L}_\bullet$ or $\mathbb{L}_\circ$, we say that a subset of $\mathbb{R}^2$ is *within* C if it is contained in the topological closure of the bounded connected component of $\mathbb{R}^2 \setminus C$. A sub-graph $G = (V, E)$ of $\mathbb{L}_\bullet$ is called an $\mathbb{L}_\bullet$-domain of the *first kind* if $E = \mathbb{E}_V$ and V is given by the vertices within a simple circuit in $\mathbb{L}_\bullet$. We define $\mathbb{L}_\circ$-domains of the first kind in the same manner. We say that a sub-graph $G = (V, E)$ of $\mathbb{L}_\bullet$ is a domain of the *second kind* if its dual $(\mathbb{V}_{*E}, *E)$ is an $\mathbb{L}_\circ$-domain of the first kind. It is called an $\mathbb{L}_\bullet$-domain if it is an $\mathbb{L}_\bullet$-domain of the first or second kind. We define $\mathbb{L}_\circ$-domains (of the second kind) analogously; see Figure 11.

Given a sub-graph $G = (V, E)$ of $\mathbb{L}_\bullet$ with dual $G^* = (\mathbb{V}_{*E}, *E)$, define Δ_G as the set of vertices of degree 4 in G and G^*, that is,

$$\Delta_G := \{i \in V : |\partial^{\text{edge}}_G\{i\}| = 4\} \cup \{u \in \mathbb{V}_{*E} : |\partial^{\text{edge}}_{G^*}\{u\}| = 4\}.$$

Observe that, if G is an $\mathbb{L}_\bullet$-domain, then $\mathcal{D}_G := \Delta_G$ is an $\mathbb{L}$-domain. In this case, any tile $t \in \mathbb{L}_\diamond$ intersects $\mathcal{D}_G$ precisely if its associated edge $e_t \in \mathbb{E}^\bullet$ belongs to E, that is, $\{e_t : t \in A_{\mathcal{D}_\mathcal{G}}\} = E$; see Section 5.1.1 and Figure 11.

5.1.3. *Duality coupling with the AT model.* The ATRC measures can be sampled locally from the HF measures, which is the content of the following lemma. We build on [GP23], where a marginal of the ATRC measure was sampled. Given $\mathbf{c} > 2$, a

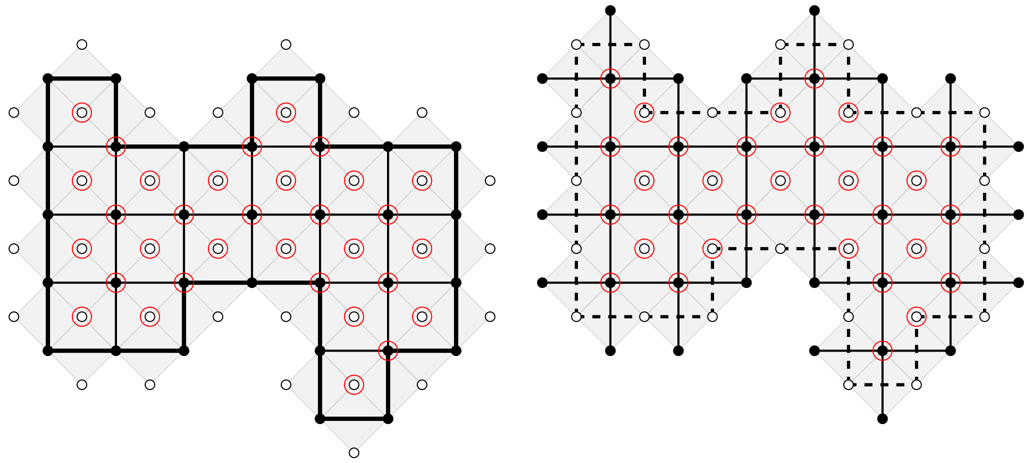

FIGURE 11. Left: a circuit in $\mathbb{L}_\bullet$ (thick black edges) and the corresponding $\mathbb{L}_\bullet$-domain of the first kind (thin and thick black edges). Right: a circuit in $\mathbb{L}_\circ$ (dashed edges) obtained by shifting the left one by $(1/2, 1/2)$ and the corresponding $\mathbb{L}_\bullet$-domain of the second kind (black edges). The tiles corresponding to the edges of the domains are shaded grey, and the vertices of the corresponding $\mathbb{L}$-domains are surrounded by red circles.

height function $h \in \mathbb{Z}^{\mathbb{L}}$, an edge $e = ij \in \mathbb{E}^\bullet$ with $e^* = uv$, and $x \in [0,1]$, define $\mathcal{X}_{\mathbf{c}}(h,e,x) \in \{(0,0),(0,1),(1,1)\}$ as follows:

$$
\begin{aligned}
&\bullet \text{ if } h(u) \neq h(v), \text{ set } \mathcal{X}_{\mathbf{c}}(h,e,x) = (1,1),\\
&\bullet \text{ if } h(i) \neq h(j), \text{ set } \mathcal{X}_{\mathbf{c}}(h,e,x) = (0,0),\\
&\bullet \text{ if } h(u) = h(v) \text{ and } h(i) = h(j), \text{ set}\\
&\qquad \mathcal{X}_{\mathbf{c}}(h,e,x) = \mathbb{1}_{[0,\frac{1}{\mathbf{c}})}(x)\cdot(1,1) + \mathbb{1}_{[\frac{1}{\mathbf{c}},\frac{2}{\mathbf{c}})}(x)\cdot(0,0) + \mathbb{1}_{[\frac{2}{\mathbf{c}},1]}(x)\cdot(0,1).
\end{aligned} \tag{22}
$$

Observe that, in order to define $\mathcal{X}_{\mathbf{c}}(h,e,x)$, it suffices to know the values of the corresponding six-vertex spins $\sigma_\bullet(i), \sigma_\bullet(j), \sigma_\circ(u), \sigma_\circ(v)$, defined in (20).

Lemma 5.1. *Let $0 < J < U$ satisfy* $\sinh 2J = e^{-2U}$ *and take* $\mathbf{c} := \coth 2J$. *Let $G = (V,E)$ be an $\mathbb{L}_\bullet$-domain, and let $\mathcal{D} := \mathcal{D}_G$ be its associated $\mathbb{L}$-domain. Let h be distributed according to $\mathsf{HF}^{0,1}_{\mathcal{D};\mathbf{c}}$, and let $(U_e)_{e\in E}$ be a sequence of i.i.d. uniform random variables on $[0,1]$, independent of h. Define $\omega_\tau, \omega_{\tau\tau'} \in \{0,1\}^{\mathbb{E}^\bullet}$ by setting, for $e \in \mathbb{E}^\bullet$,*

$$
(\omega_\tau(e), \omega_{\tau\tau'}(e)) := \begin{cases} (0,1) & \text{if } e \in \mathbb{E}^\bullet \setminus E, \\ \mathcal{X}_{\mathbf{c}}(h,e,U_e) & \text{if } e \in E. \end{cases} \tag{23}
$$

Then, the law of $(\omega_\tau, \omega_{\tau\tau'})$ is given by $\mathsf{ATRC}^{0,1}_{G;J,U}$.

Proof. Recall that a tile $t \in \mathbb{L}_\diamond$ intersects $\mathcal{D}$ precisely if its corresponding edge $e_t \in \mathbb{E}^\bullet$ belongs to E, see Figure 11. Let $(\sigma_\bullet, \sigma_\circ)$ with values in $\{\pm1\}^{\mathbb{L}_\bullet} \times \{\pm1\}^{\mathbb{L}_\circ}$ be the spin configurations corresponding to h, defined by (20). For ease of notation, we use the same symbols for the random variables and their realisations. For $\sigma_\bullet \in \{\pm1\}^{\mathbb{L}_\bullet}$, $\sigma_\circ \in \{\pm1\}^{\mathbb{L}_\circ}$, $\omega \in \{0,1\}^{\mathbb{E}^\bullet}$ and $\# \in \{0,1\}$, define

$$
\Sigma^+_{\mathcal{D}_\bullet}(\sigma_\bullet) := \mathbb{1}_{\sigma_\bullet \equiv 1 \text{ on } \mathbb{L}_\bullet \setminus \mathcal{D}_\bullet}, \quad \Sigma^+_{\mathcal{D}_\circ}(\sigma_\circ) := \mathbb{1}_{\sigma_\circ \equiv 1 \text{ on } \mathbb{L}_\circ \setminus \mathcal{D}_\circ}, \quad \Omega^\#_E(\omega) = \mathbb{1}_{\omega \equiv \# \text{ on } \mathbb{E}^\bullet \setminus E}.
$$

Then, the law of the quadruple $(\sigma_\bullet, \sigma_\circ, \omega_\tau, \omega_{\tau\tau'})$ can be written as follows:

$$(\sigma_\bullet, \sigma_\circ, \omega_\tau, \omega_{\tau\tau'}) \propto \mathbb{1}_{\mathrm{ice}}(\sigma_\bullet, \sigma_\circ)\, \Sigma^+_{\mathcal{D}_\bullet}(\sigma_\bullet)\, \Sigma^+_{\mathcal{D}_\circ}(\sigma_\circ)\, \mathbb{1}_{\sigma_\bullet \sim \omega_{\tau\tau'}}\, \mathbb{1}_{\sigma_\circ \sim \omega^*_\tau} \cdot (\mathbf{c}-2)^{|(\omega_{\tau\tau'} \setminus \omega_\tau) \cap E|}\, \mathbb{1}_{\omega_\tau \subseteq \omega_{\tau\tau'}}\, \Omega^0_E(\omega_\tau)\, \Omega^1_E(\omega_{\tau\tau'}),$$

where $\sigma_\bullet \sim \omega_{\tau\tau'}$ denotes the event that $\sigma_\bullet$ is constant on edges in $\omega_{\tau\tau'}$ and similarly for $\sigma_\circ \sim \omega^*_\tau$. Observe that $\sigma_\bullet \sim \omega_{\tau\tau'}$, $\sigma_\circ \sim \omega^*_\tau$ and $\omega_\tau \subseteq \omega_{\tau\tau'}$ imply that $(\sigma_\bullet, \sigma_\circ)$ satisfies the ice rule. Summing over $\sigma_\bullet$ and $\sigma_\circ$, we obtain the law of $(\omega_\tau, \omega_{\tau\tau'})$:

$$(\omega_\tau, \omega_{\tau\tau'}) \propto 2^{\kappa_V(\omega_{\tau\tau'}) + \kappa_{V'}(\omega^*_\tau)}\, \Omega^0_E(\omega_\tau)\, \Omega^1_E(\omega_{\tau\tau'})\, \mathbb{1}_{\omega_\tau \subseteq \omega_{\tau\tau'}}\, (\mathbf{c}-2)^{|(\omega_{\tau\tau'} \setminus \omega_\tau) \cap E|},$$

where $V' := \mathbb{V}_{*E}$, and where we used that there exist $2^{\kappa_V(\omega_{\tau\tau'})-1}$ spin configurations $\sigma_\bullet \sim \omega_{\tau\tau'}$ with $\Sigma^+_{\mathcal{D}_\bullet}(\sigma_\bullet) = 1$, and similarly for $\sigma_\circ \sim \omega^*_\tau$. By planar duality, $\kappa_{V'}(\omega^*_\tau) = \kappa_V(\omega_\tau) + |\omega_\tau \cap E| + \mathrm{const}(G)$. Since $\mathrm{w}_\tau, \mathrm{w}_{\tau\tau'}$ from (4) satisfy $\mathrm{w}_\tau = 2$ and $\mathrm{w}_{\tau\tau'} = \mathbf{c}-2$, the proof is complete. □

5.1.4. *Baxter–Kelland–Wu coupling.* In Section 4.3, we described the BKW coupling by sampling the FK-percolation from the six-vertex model. We now focus on the reverse direction of the BKW. Fix $q > 4$ and $\beta = \beta_c(q) = \ln(1+\sqrt{q})$, and let $\lambda, \mathbf{c}, \mathbf{c}_\mathrm{b}$ be as in (7) and (8). Recall the loop representation of FK-percolation and the representations of the six-vertex model introduced in Section 4.2.

Let $\mathcal{D}$ be an even $\mathbb{L}$-domain, and set

$$E := *\mathbb{E}_{\mathcal{D}_\circ} \subset \mathbb{E}^\bullet, \quad \Lambda := \mathbb{V}_E \subset \mathbb{L}_\bullet, \quad \text{and} \quad G := (\Lambda, E). \tag{24}$$

The following result [BKW76] is classical. It may be proved through the arguments presented in the proof of Lemma 4.1; see [GP23, Theorem 7] for a proof in a similar setting. We will construct a height function from a loop configuration by increasing or decreasing the height whenever we cross a loop, which is the same as orienting the loops as in Section 4.3. We chose the first option for the sake of brevity.

Proposition 5.2. *Let ω be a random element of $\{0,1\}^{\mathbb{E}^\bullet}$ distributed according to $\mathsf{FK}^{\mathrm{w}}_{G;\beta,q}$. Consider the corresponding loop configuration* $\mathrm{loop}(\omega)$*, and define a height function h as follows:*

- ***H1*** *Set $h(i) = 0$ for $i \in \mathbb{L}_\bullet \setminus \mathcal{D}$, and $h(u) = 1$ for $u \in \mathbb{L}_\circ \setminus \mathcal{D}$.*
- ***H2*** *Assign constant values to clusters of ω and ω^* in $\mathcal{D}$ by **decreasing** the height by 1 with probability $e^\lambda/\sqrt{q}$ and **increasing** the height by 1 with probability $e^{-\lambda}/\sqrt{q}$ when crossing a loop from outside, independently for every loop.*

The height function h is distributed according to $\mathsf{HF}^{0,1}_{\mathcal{D};\mathbf{c},\mathbf{c}_\mathrm{b}}$.

The above procedure also works for odd $\mathbb{L}$-domains with the difference that one has to take $\omega \sim \mathsf{FK}^{\mathrm{f}}_{\mathcal{D}_\bullet;p,q}$, and **H2** must be replaced by

- **H2'** Assign constant values to clusters of ω and ω^* in $\mathcal{D}$ by **decreasing** the height by 1 with probability $e^\lambda/\sqrt{q}$ and **increasing** the height by 1 with probability $e^{-\lambda}/\sqrt{q}$ when crossing a loop from outside, independently for every loop.

5.1.5. *Input from* [GP23]. For the whole section, fix $q > 4$ and $\beta = \beta_c(q) = \ln(1+\sqrt{q})$, and let $\lambda, \mathbf{c}, \mathbf{c}_\mathrm{b}$ be as in (7) and (8). The following proposition is a consequence of [GP23, Proposition 6.1 and Lemma 6.2] and their proofs.

Proposition 5.3. *For any sequence of even or odd $\mathbb{L}$-domains $\mathcal{D}_k \nearrow \mathbb{L}$, the measures $\mathsf{HF}^{0,1}_{\mathcal{D}_k;\mathbf{c},\mathbf{c}_\mathrm{b}}$ converge to some $\mathsf{HF}^{0,1}_{\mathbf{c}}$ which is independent of the sequence $(\mathcal{D}_k)$. Moreover, the limiting measure $\mathsf{HF}^{0,1}_{\mathbf{c}}$ can be constructed in either of the following two ways:*

(i) *Sample* $\omega \in \{0,1\}^{\mathbb{E}^\bullet}$ *according to* $\mathsf{FK}^{\mathrm{w}}_{\beta,q}$. *Set* $h=0$ *on the unique infinite cluster of* ω*, and sample* h *elsewhere according to* **H2** *in Section 5.1.4.*
(ii) *Sample* $\omega \in \{0,1\}^{\mathbb{E}^\bullet}$ *according to* $\mathsf{FK}^{\mathrm{f}}_{\beta,q}$. *Set* $h=1$ *on the unique infinite cluster of* ω^**, and sample* h *elsewhere according to* **H2'** *in Section 5.1.4.*

The following lemma is a slight generalisation of [GP23, Eq. (28)] and can be proved in exactly the same manner.

Lemma 5.4. *Let* $\mathcal{D}$ *be an* $\mathbb{L}$*-domain, and let* δ *be a set of boundary tiles of* $\mathcal{D}$*. Define* $\mathcal{D}^{\mathrm{even}} = \mathcal{D} \setminus (\partial^{\mathrm{in}}\mathcal{D})_\bullet$ *and* $\mathcal{D}^{\mathrm{odd}} = \mathcal{D} \setminus (\partial^{\mathrm{in}}\mathcal{D})_\circ$*, where the boundaries are taken in* $\mathbb{L}$*. Then,* $\mathcal{D}^{\mathrm{even}}$ *and* $\mathcal{D}^{\mathrm{odd}}$ *are disjoint unions of even and odd domains in* $\mathbb{L}$*, respectively. Moreover, the following stochastic ordering of measures holds:*

$$\mathsf{HF}^{0,1}_{\mathcal{D}^{\mathrm{even}};\mathbf{c},\mathbf{c}_{\mathrm{b}}} \le_{\mathrm{st}} \mathsf{HF}^{0,1}_{\mathcal{D},\delta;\mathbf{c},\mathbf{c}_{\mathrm{b}}} \le_{\mathrm{st}} \mathsf{HF}^{0,1}_{\mathcal{D}^{\mathrm{odd}};\mathbf{c},\mathbf{c}_{\mathrm{b}}}.$$

This lemma readily implies that $\mathsf{HF}^{0,1}_{\mathcal{D}_k,\delta_k;\mathbf{c},\mathbf{c}_{\mathrm{b}}}$ (in particular $\mathsf{HF}^{0,1}_{\mathcal{D}_k;\mathbf{c}}$) converges to the same limit, no matter which sequences of $\mathcal{D}_k$ and δ_k we chose.

5.2. **Relaxation of height function measures.** The aim of this section is to prove a relaxation statement for the six-vertex height function measures with modified weight $\mathbf{c}_{\mathrm{b}}$ on arbitrary boundary tiles, in particular for the measure with unmodified weight $\mathbf{c}$ on all tiles. For the whole section, fix $q>4$ and $\beta=\beta_c(q)=\ln\bigl(1+\sqrt{q}\bigr)$, and let $\lambda, \mathbf{c}, \mathbf{c}_{\mathrm{b}}$ be as in (7) and (8).

Given a measure HF on $\mathbb{Z}^{\mathbb{L}}$ and $\Delta \subset \mathbb{L}$, define $\mathsf{HF}|_\Delta$ as its marginal on $\mathbb{Z}^\Delta$.

Proposition 5.5. *There exist constants* $c,\alpha>0$ *such that, for every finite* $\Delta \subset \mathbb{L}$*, every* $\mathbb{L}$*-domain* $\mathcal{D}$ *that contains* Δ*, and every set* $\delta \subset \mathbb{L}_\diamond$ *of boundary tiles of* $\mathcal{D}$*,*

$$\mathrm{d_{TV}}\left(\mathsf{HF}^{0,1}_{\mathcal{D},\delta;\mathbf{c},\mathbf{c}_{\mathrm{b}}}|_\Delta, \mathsf{HF}^{0,1}_{\mathbf{c}}|_\Delta)\right) < c\,|\Delta|\,(\mathrm{diam}(\Delta)+\mathrm{d}_\infty(\Delta,\mathcal{D}^c))^2\, e^{-\alpha\,\mathrm{d}_\infty(\Delta,\mathcal{D}^c)}.$$

The statement on even or odd $\mathbb{L}$-domains and with modified weight $\mathbf{c}_{\mathrm{b}}$ on all boundary tiles can be proven in the same way as [ADG24, Proposition 4.2], which is slightly less general. We provide only the statement.

Lemma 5.6. *There exist constants* $c,\alpha>0$ *such that, for every finite* $\Delta\subset\mathbb{L}$ *and every even or odd* $\mathbb{L}$*-domain* $\mathcal{D}$ *that contains* Δ*,*

$$\mathrm{d_{TV}}\left(\mathsf{HF}^{0,1}_{\mathcal{D};\mathbf{c},\mathbf{c}_{\mathrm{b}}}|_\Delta, \mathsf{HF}^{0,1}_{\mathbf{c}}|_\Delta\right) < c\,(\mathrm{diam}(\Delta)+\mathrm{d}_\infty(\Delta,\mathcal{D}^c))\, e^{-\alpha\,\mathrm{d}_\infty(\Delta,\mathcal{D}^c)}.$$

Before deducing Proposition 5.5 from the above and the stochastic ordering of height function measures, Lemma 5.4, we need a general lemma.

Lemma 5.7. [HS22, Lemma 2.16] *Let* μ *and* ν *be two probability measures on a finite* totally ordered *set* S *with* $\mu \le_{\mathrm{st}} \nu$*. Let* $\mathbf{P}$ *be a monotone coupling of* $X\sim\mu$ *and* $Y\sim\nu$*. Then,*

$$\mathbf{P}(X\neq Y) \le (|S|-1)\,\mathrm{d_{TV}}(\mu,\nu).$$

Proof. Identify S with $\{0,\dots,|S|-1\}$. Let $\mathbf{P}_{\mathrm{op}}$ be an optimal coupling of X and Y, that is, $\mathbf{P}_{\mathrm{op}}(X\neq Y)=\mathrm{d_{TV}}(\mu,\nu)$. Since $\mathbf{P}(X\le Y)=1$, we have

$$\begin{aligned}\mathbf{P}(X\neq Y) = \mathbf{P}(Y-X\ge 1) &\le \mathbf{E}[Y-X]\\ &= \mathbf{E}_{\mathrm{op}}[Y-X] \le (|S|-1)\,\mathbf{P}_{\mathrm{op}}(X\neq Y),\end{aligned}$$

where we applied Markov's inequality. □

Proof of Proposition 5.5. Let $\Delta \subset \mathbb{L}$ be finite, let $\mathcal{D}$ be an $\mathbb{L}$-domain, and let $\delta \subset \mathbb{L}_\diamond$ be a set of boundary tiles of $\mathcal{D}$. Assume without loss of generality that $\Delta \subseteq \mathcal{D} \setminus \partial^{\text{in}}\mathcal{D}$. Let $\mathcal{D}^{\text{even}}$ and $\mathcal{D}^{\text{odd}}$ be as in Lemma 5.4, and recall the stochastic domination statement therein. Consider a monotone coupling $\mathbf{P}$ of $h^- \sim \mathsf{HF}^{0,1}_{\mathcal{D}^{\text{even}};\mathbf{c},\mathbf{c}_\text{b}}$, $h \sim \mathsf{HF}^{0,1}_{\mathcal{D},\delta;\mathbf{c},\mathbf{c}_\text{b}}$ and $h^+ \sim \mathsf{HF}^{0,1}_{\mathcal{D}^{\text{odd}};\mathbf{c},\mathbf{c}_\text{b}}$, that is,

$$\mathbf{P}(h^- \le h \le h^+) = 1.$$

Take any vertex $v \in \Delta$ and note that, deterministically, the height functions at v take values in $[-m, m]$, where $m := 2(\text{diam}(\Delta) + \text{d}_\infty(\Delta, \mathcal{D}^c))$. By Lemma 5.7, we have

$$\begin{aligned} \mathbf{P}(h^-(v) \neq h(v)) \le \mathbf{P}(h^-(v) \neq h^+(v)) &\le 2m\, \text{d}_{\text{TV}}\big(\mathsf{HF}^{0,1}_{\mathcal{D}^{\text{even}};\mathbf{c},\mathbf{c}_\text{b}}|_\Delta, \mathsf{HF}^{0,1}_{\mathcal{D}^{\text{odd}};\mathbf{c},\mathbf{c}_\text{b}}|_\Delta\big) \qquad (25)\\ &\le 2m\left(\text{d}_{\text{TV}}\big(\mathsf{HF}^{0,1}_{\mathcal{D}^{\text{even}};\mathbf{c},\mathbf{c}_\text{b}}|_\Delta, \mathsf{HF}^{0,1}_{\mathbf{c}}|_\Delta\big) + \text{d}_{\text{TV}}\big(\mathsf{HF}^{0,1}_{\mathbf{c}}|_\Delta, \mathsf{HF}^{0,1}_{\mathcal{D}^{\text{odd}};\mathbf{c},\mathbf{c}_\text{b}}|_\Delta\big)\right). \end{aligned}$$

Now, $\mathcal{D}^{\text{even}}$ is a disjoint union of even $\mathbb{L}$-domains $\mathcal{D}_1, \dots, \mathcal{D}_n$ with $\partial^{\text{ex}}\mathcal{D}_i \cap \mathcal{D}_j = \varnothing$ for any $i \neq j$. There exists an i with $\Delta \subseteq \mathcal{D}_i$ and $\text{d}_\infty(\Delta, \mathcal{D}_i^c) \ge \text{d}_\infty(\Delta, \mathcal{D}^c) - 1$. Moreover, the definition of the height function measures (21) readily implies that, for any event $A \subset \mathbb{Z}^{\mathbb{L}}$ depending only on values in Δ, we have $\mathsf{HF}^{0,1}_{\mathcal{D}^{\text{even}};\mathbf{c},\mathbf{c}_\text{b}}(A) = \mathsf{HF}^{0,1}_{\mathcal{D}_i;\mathbf{c},\mathbf{c}_\text{b}}(A)$. Similar reasoning applies to $\mathcal{D}^{\text{odd}}$. By Lemma 5.6, there exist $c, \alpha > 0$, such that the right-hand side of (25) is bounded by $c\, m^2\, e^{-\alpha\, \text{d}_\infty(\Delta, \mathcal{D}^c)}$. The union bound gives

$$\text{d}_{\text{TV}}\left(\mathsf{HF}^{0,1}_{\mathcal{D}^{\text{even}};\mathbf{c},\mathbf{c}_\text{b}}|_\Delta, \mathsf{HF}^{0,1}_{\mathcal{D},\delta;\mathbf{c},\mathbf{c}_\text{b}}|_\Delta\right) \le \mathbf{P}(h^-|_\Delta \neq h|_\Delta) \le \sum_{v \in \Delta} \mathbf{P}(h^-(v) \neq h(v)).$$

Another application of the triangle inequality and of Lemma 5.6 finishes the proof. □

Remark 5.8. *Using that h^-, h^+ have uniformly bounded second moment (localisation) and adapting Lemma 5.7 allow to remove the square of* $(\text{diam}(\Delta) + \text{d}_\infty(\Delta, \mathcal{D}^c))^2$ *in the bound in Proposition 5.5.*

5.3. **Relaxation of ATRC measures.** The proof of Proposition 3.2 is based on Proposition 5.11 and another one below, which is a consequence of Proposition 5.5.

The following proposition establishes exponential relaxation of the ATRC measures on domains and with '0, 1' boundary conditions. Once Proposition 3.2 is proven, it can be concluded that the limit coincides with the unique ATRC Gibbs measure. Recall the definition (22) of $\mathcal{X}_\mathbf{c}$.

Proposition 5.9. *Let $0 < J < U$ satisfy $\sinh 2J = e^{-2U}$. There exists a measure $\mathsf{ATRC}_{J,U}$ on $\{0,1\}^{\mathbb{E}^\bullet} \times \{0,1\}^{\mathbb{E}^\bullet}$ and constants $c, \alpha > 0$ such that, for any $\mathbb{L}_\bullet$-domain $G = (V, E)$ and any $F \subseteq E$,*

$$\begin{aligned} \text{d}_{\text{TV}}\big(\mathsf{ATRC}^{0,1}_{G;J,U}[(\omega_\tau|_F, \omega_{\tau\tau'}|_F) \in \cdot\,], \mathsf{ATRC}_{J,U}[(\omega_\tau|_F, \omega_{\tau\tau'}|_F) \in \cdot\,]\big)& \\ < c\,|\mathbb{V}_F|\,(\text{diam}(\mathbb{V}_F) + \text{d}_\infty(\mathbb{V}_F, V^c))^2\, e^{-\alpha\, \text{d}_\infty(\mathbb{V}_F, V^c)},& \end{aligned}$$

where diam *denotes the diameter with respect to d_∞. Furthermore, the measure $\mathsf{ATRC}_{J,U}$ is constructed as follows. Let $\mathbf{c} = \coth 2J$, let h be distributed according to $\mathsf{HF}^{0,1}_{\mathbf{c}}$, and let $(U_e)_{e \in \mathbb{E}^\bullet}$ be a sequence of i.i.d. uniform random variables on $[0,1]$, independent of h. The measure $\mathsf{ATRC}_{J,U}$ is given by the law of $(\omega_\tau, \omega_{\tau\tau'}) \in \{0,1\}^{\mathbb{E}^\bullet} \times \{0,1\}^{\mathbb{E}^\bullet}$ defined by*

$$(\omega_\tau(e), \omega_{\tau\tau'}(e)) := \mathcal{X}_\mathbf{c}(h, e, U_e), \;\; e \in \mathbb{E}^\bullet.$$

Remark 5.10. *It is possible to improve the bound in Proposition 5.9 and remove the square in* $(\text{diam}(\mathbb{V}_F) + \text{d}_\infty(\mathbb{V}_F, V^c))^2$*, see Remark 5.8.*

The above proposition follows from the analogous statement for the height function measures, Proposition 5.5, and the fact that the ATRC measures can be sampled locally from these measures, which is the content of Lemma 5.1.

Proof of Proposition 5.9. Let $0 < J < U$ satisfy $\sinh 2J = e^{-2U}$ and $\mathbf{c} = \coth 2J$. Let $G = (V, E)$ be an $\mathbb{L}_\bullet$-domain and $\mathcal{D} := \mathcal{D}_G$ be its associated $\mathbb{L}$-domain. Recall that a tile $t \in \mathbb{L}_\diamond$ intersects $\mathcal{D}$ precisely if its associated primal edge $e_t \in \mathbb{E}^\bullet$ belongs to E, see Figure 11. Let $F \subset E$, and set $\Delta := \mathbb{V}_F \cup \mathbb{V}_{*F}$.

Take $h \sim \mathsf{HF}^{0,1}_{\mathcal{D};\mathbf{c}}$ and $h' \sim \mathsf{HF}^{0,1}_{\mathbf{c}}$, and consider an optimal coupling of $h|_\Delta$ and $h'|_\Delta$. Let $(U_e)_{e\in\mathbb{E}^\bullet}$ be a sequence of i.i.d. uniform random variables on $[0,1]$, independent of (h,h'). Denote the corresponding probability measure by $\mathbf{P}$. Define $\omega_\tau, \omega_{\tau\tau'} \in \{0,1\}^{\mathbb{E}^\bullet}$ from h and (U_e) as in (23). Define $\omega'_\tau, \omega'_{\tau\tau'}$ from h' and (U_e) by setting $(\omega'_\tau(e), \omega'_{\tau\tau'}(e)) = \mathcal{X}_{\mathbf{c}}(h', e, U_e)$ for $e \in \mathbb{E}^\bullet$. Then, clearly

$$(\omega_\tau|_F, \omega_{\tau\tau'}|_F) \neq (\omega'_\tau|_F, \omega'_{\tau\tau'}|_F) \quad \text{implies} \quad h|_\Delta \neq h'|_\Delta.$$

Therefore, by the definition of an optimal coupling and by Proposition 5.5,

$$\begin{aligned}\mathbf{P}\big((\omega_\tau|_F, \omega_{\tau\tau'}|_F) \neq (\omega'_\tau|_F, \omega'_{\tau\tau'}|_F)\big) &< c\,|\Delta|\,(\mathrm{diam}(\Delta) + \mathrm{d}_\infty(\Delta, \mathcal{D}^c))^2\, e^{-\alpha\, \mathrm{d}_\infty(\Delta,\mathcal{D}^c)} \\ &< c'\,|F|\,(\mathrm{diam}(\mathbb{V}_F) + \mathrm{d}_\infty(\mathbb{V}_F, V^c))^2\, e^{-\alpha\, \mathrm{d}_\infty(\mathbb{V}_F,V^c)},\end{aligned}$$

for some $c, c', \alpha > 0$. By Lemma 5.1, the law of $(\omega_\tau, \omega_{\tau\tau'})$ is given by $\mathsf{ATRC}^{0,1}_{G;J,U}$. Finally, denote the law of $(\omega'_\tau, \omega'_{\tau\tau'})$ by $\mathsf{ATRC}_{J,U}$, and the proof is complete. □

5.4. **Proof of Proposition 3.2.** Finally, we derive Proposition 3.2 from Proposition 5.9 and the following one that we import from [ADG24].

Proposition 5.11. [ADG24, Proposition 1.1] *Let $0 < J < U$ satisfy $\sinh 2J = e^{-2U}$. There exists $c > 0$ such that, for every $n \geq 1$,*

$$\mathsf{ATRC}^{1,1}_{\Lambda_n;J,U}(0 \xleftrightarrow{\omega_\tau} \partial^{\mathrm{ex}}\Lambda_n) < e^{-cn}.$$

Proof of Proposition 3.2. Let $\sigma \in \{\tau, \tau\tau'\}$. Assume without loss of generality that $n = 2k$ is even (the other case is treated similarly). We proceed in two steps.
Claim 1. There exists $\alpha > 0$ such that

$$\mathsf{ATRC}^{1,1}_{\Lambda_n;J,U}[\omega_\sigma(e)] - \mathsf{ATRC}^{0,1}_{\Lambda_n;J,U}[\omega_\sigma(e)] \leq e^{-\alpha n}.$$

Let $(\omega_\tau, \omega_{\tau\tau'})$ be distributed according to $\mathsf{ATRC}^{1,1}_{\Lambda_n;J,U}$. Define $\mathcal{C}$ to be the outermost (dual) circuit in ω^*_τ that surrounds Λ_k and is contained in Λ_{2k} if it exists, otherwise set $\mathcal{C} = \varnothing$. By exponential decay of connection probabilities in ω_τ, Proposition 5.11, there exist $c, \alpha > 0$ such that

$$\begin{aligned}\mathsf{ATRC}^{1,1}_{\Lambda_n;J,U}[\omega_\sigma(e)] &\leq \mathsf{ATRC}^{1,1}_{\Lambda_n;J,U}\left[\omega_\sigma(e)\,\middle|\,\mathcal{C} \neq \varnothing\right] + \mathsf{ATRC}^{1,1}_{\Lambda_n;J,U}[\Lambda_k \xleftrightarrow{\omega_\tau} \partial^{\mathrm{in}}\Lambda_{2k}] \\ &\leq \mathsf{ATRC}^{1,1}_{\Lambda_n;J,U}\left[\omega_\sigma(e)\,\middle|\,\mathcal{C} \neq \varnothing\right] + cke^{-\alpha k},\end{aligned}$$

where we also used the strong FKG and spatial Markov properties of the ATRC measures (Lemma 3.1 and (SMP) in Section 3). Now, again by the strong FKG and (SMP) and by exponential relaxation, Proposition 5.9, there exist $c', \alpha' > 0$ such that

$$\begin{aligned}
&\mathsf{ATRC}^{1,1}_{\Lambda_n;J,U}\left[\omega_\sigma(e)\,\middle|\,\mathcal{C}\neq\varnothing\right]\\
&\qquad=\sum_{C}\mathsf{ATRC}^{1,1}_{\Lambda_n;J,U}\left[\omega_\sigma(e)\,\middle|\,\mathcal{C}=C\right]\mathsf{ATRC}^{1,1}_{\Lambda_n;J,U}\left[\mathcal{C}=C\,\middle|\,\mathcal{C}\neq\varnothing\right]\\
&\qquad\leq\sum_{C}\mathsf{ATRC}^{0,1}_{G_C;J,U}[\omega_\sigma(e)]\;\mathsf{ATRC}^{1,1}_{\Lambda_n;J,U}\left[\mathcal{C}=C\,\middle|\,\mathcal{C}\neq\varnothing\right]\\
&\qquad\leq\mathsf{ATRC}^{0,1}_{\Lambda_n;J,U}\left[\omega_\sigma(e)\right]+c'k^2e^{-\alpha' k},
\end{aligned}$$

where the summation is over all realisations C of $\mathcal{C}$, and where G_C is the largest $\mathbb{L}_\bullet$-domain of the first kind within C that contains Λ_k. This proves Claim 1.

Claim 2. There exists $\alpha>0$ such that

$$\mathsf{ATRC}^{0,1}_{\Lambda_n;J,U}[\omega_\sigma(e)]-\mathsf{ATRC}^{0,0}_{\Lambda_n;J,U}[\omega_\sigma(e)]\leq e^{-\alpha n}.$$

We use the same strategy as in Claim 1. Indeed, let $(\tilde\omega_\tau,\tilde\omega_{\tau\tau'})$ be distributed according to $\mathsf{ATRC}^{0,0}_{\Lambda_n;J,U}$. Since, by self-duality (6), $\tilde\omega^*_{\tau\tau'}$ has the same law as ω_τ (but on the dual graph of Λ_n) from Claim 1, Proposition 5.11 allows to find a circuit in $\tilde\omega_{\tau\tau'}$ that surrounds Λ_k and is contained in Λ_{2k}. For a realisation C of an outermost such circuit, consider the largest $\mathbb{L}_\bullet$-domain of the second kind within C that contains Λ_k. The remainder of the argument is analogous to Claim 1. □

5.5. **Ratio weak mixing: proof of Theorem 4.** We first derive the *exponential* weak mixing property from the single edge relaxation, Proposition 3.2. Given $F\subset\mathbb{E}^\bullet$ and a measure ATRC on $\{0,1\}^{\mathbb{E}^\bullet}\times\{0,1\}^{\mathbb{E}^\bullet}$, we write $\mathsf{ATRC}|_F$ for its marginal on $\{0,1\}^F\times\{0,1\}^F$.

Theorem 7. *Let $0<J<U$ satisfy $\sinh 2J=e^{-2U}$. There exists $c>0$ such that, for any finite sub-graph $G=(V,E)$ of $\mathbb{L}_\bullet$ and any $F\subset E$,*

$$\sup_{\xi_\tau,\xi_{\tau\tau'},\xi'_\tau,\xi'_{\tau\tau'}}\mathrm{d_{TV}}\Big(\mathsf{ATRC}^{\xi_\tau,\xi_{\tau\tau'}}_{G;J,U}|_F,\mathsf{ATRC}^{\xi'_\tau,\xi'_{\tau\tau'}}_{G;J,U}|_F\Big)\leq 2\sum_{e\in F}e^{-c\,\mathrm{d}_\infty(e,V^c)},$$

where d_∞ is the distance induced by the L^∞ norm.

As a consequence, the measures $\mathsf{ATRC}^{\xi_\tau,\xi_{\tau\tau'}}_{G;J,U}$ converge (as $G\nearrow\mathbb{L}_\bullet$) to a limit measure $\mathsf{ATRC}_{J,U}$, which is independent of the choice of boundary conditions $\xi_\tau,\xi_{\tau\tau'}$. Furthermore, the limit $\mathsf{ATRC}_{J,U}$ is the unique ATRC *Gibbs* measure.

Proof. By strong positive association, Lemma 3.1, the supremum is attained for $\xi_\tau=\xi_{\tau\tau'}=1$ and $\xi'_\tau=\xi'_{\tau\tau'}=0$. Let Ψ be a monotone coupling of $\mathsf{ATRC}^{1,1}_{G;J,U}$ and $\mathsf{ATRC}^{0,0}_{G;J,U}$, and let $(X,Y)=\big((X_\tau,X_{\tau\tau'}),(Y_\tau,Y_{\tau\tau'})\big)\sim\Psi$. Then,

$$\begin{aligned}
\mathrm{d_{TV}}\Big(\mathsf{ATRC}^{1,1}_{G;J,U}|_F,\mathsf{ATRC}^{0,0}_{G;J,U}|_F\Big)&\leq\Psi(X|_F\neq Y|_F)\\
&\leq\Psi(X_\tau|_F\neq Y_\tau|_F)+\Psi(X_{\tau\tau'}|_F\neq Y_{\tau\tau'}|_F).
\end{aligned}$$

Now, for $\sigma\in\{\tau,\tau\tau'\}$,

$$\begin{aligned}
\Psi(X_\sigma|_F\neq Y_\sigma|_F)&\leq\sum_{e\in F}\Psi(X_\sigma(e)\neq Y_\sigma(e))=\sum_{e\in F}\Psi(X_\sigma(e)>Y_\sigma(e))\\
&=\sum_{e\in F}\left(\Psi(X_\sigma(e)=1)-\Psi(Y_\sigma(e)=1)\right)\leq\sum_{e\in F}e^{-c\mathrm{d}_\infty(e,V^c)}
\end{aligned}$$

for some $c>0$ by Proposition 3.2, where we also used strong positive association and the spatial Markov property of the ATRC measures. □

A less trivial consequence is the *ratio* weak mixing property, Theorem 4. This follows from the study performed in [Ale98], relying on Theorem 7 and Proposition 5.11.

Proof of Theorem 4. This is a direct application of [Ale98, Section 5]. The reasoning there requires two properties of the measure. The first is exponential weak mixing, which is the content of Theorem 7. The second is the admittance of exponentially bounded controlling regions in the sense of [Ale98], and we argue its validity as follows.

Let $(\omega_\tau, \omega_{\tau\tau'}) \sim \mathsf{ATRC}^{\xi_\tau, \xi_{\tau\tau'}}_{G;J,U}$, fix $F \subset E$, and define

$$H(F) = \{e \in E \setminus F : e \xleftrightarrow{\omega_\tau} \mathbb{V}_F \text{ or } e^* \xleftrightarrow{\omega^*_{\tau\tau'}} \mathbb{V}_{*F}\}.$$

Conditionally on the states of the edges in $H(F)$, the values of the field in F and $E \setminus (F \cup H(F))$ are independent. Moreover, the probability that $e \in E \setminus F$ belongs to $H(F)$ decays exponentially in $\mathrm{d}_\infty(e, \mathbb{V}_F)$ by uniform exponential decay in ω_τ and in the dual of $\omega_{\tau\tau'}$. The former is the content of Proposition 5.11 and the latter follows from self-duality. One can then apply [Ale98, Section 5] to obtain the desired claim. □

6. Strong mixing

In this section we push the mixing properties derived in the previous section to strong mixing properties for finite volume ATRC measures.

6.1. **Blocked model and setup.** Let $l \geq 1$ be a fixed large number (it has to be taken large as a function of J, U only) and let $\Lambda_l(x) = x + \{-l, \dots, l\}^2$, (with $\mathrm{e}_1 = (1,0)$, $\mathrm{e}_2 = (0,1)$)

$$E_l(x) = \{(y, y + \mathrm{e}_1), (y, y + \mathrm{e}_2) : \ y \in \Lambda_l(x)\}.$$

Let $\Gamma_l = ((2l+1)\mathbb{Z})^2$. Define the bocked model: for $a, b \in \{(0,0),(0,1),(1,1)\}^{\mathbb{E}}$,

$$\phi_i = \phi_i(a,b) = \big((a_e, b_e)\big)_{e \in E_l(i)}.$$

It is bijective mapping. For $\Lambda \subset \Gamma_l$, denote $E_l(\Lambda) = \bigcup_{i \in \Lambda} E_l(i)$.

For $\Lambda \subset \Gamma_l$

$$\partial^{\mathrm{in}} \Lambda = \{x \in \Lambda : \ \exists y \in \Lambda^c, \|x - y\|_\infty = l\}.$$

For any $F \subset \mathbb{E}$ finite, $(a,b) \in \{(0,0),(0,1),(1,1)\}^{\mathbb{E}}$, $\Lambda \subset \Gamma_l$ finite, $F' \subset \partial^{\mathrm{in}} F$, introduce a probability measure on the blocked configurations:

$$\mathrm{P}^{F,a,b,F'}_\Lambda(x) = \begin{cases} \mathsf{ATRC}^{a,b}_F(\phi^{-1}(x) \,|\, \omega_\tau(e) = 0 \,\forall e \in F') & \text{if } (\partial^{\mathrm{ex}} F \cup F') \subset E_l(\partial^{\mathrm{in}} \Lambda), \\ \mathsf{ATRC}^{a,b}_{E_l(\Lambda)}(\phi^{-1}(x)) & \text{else.} \end{cases}$$

This is the setup of [Ott25]: the model is defined on the sites of $\mathbb{Z}^2$, which is the same as Γ_l. We now have to say what is the set corresponding to the index set T of [Ott25]. The index will give the boundary conditions (a,b), the volume of the real (ATRC) model F, and what subset of the boundary, F', is conditioned to be 0 in ω_τ. We restrict which boundary conditions and F' are allowed in terms of the volume finite volume F (and which volume are allowed).

The condition will be as follows: say that the measure $\mathsf{ATRC}^{a,b}_F(\ |\omega_\tau(e) = 0 \,\forall e \in F')$ has the *l-path decoupling property* if for any simple closed path γ in Γ_l, and configurations

$$\xi = (\xi_\tau, \xi_{\tau\tau'}) \in \{(0,0),(0,1),(1,1)\}^{E_l(\gamma)}$$

such that ξ contains both a simple path of edges open in $\xi_{\tau\tau'}$ surrounding $E_l(\mathring{\gamma})$ and a simple open path of dual edges open in ξ^*_τ, with $\mathring{\gamma}$ the set of sites of Γ_l surrounded by γ but not in γ, one has that if

$$(\omega_\tau, \omega_{\tau\tau'}) \sim \mathsf{ATRC}^{a,b}_F\big(\ \mid \omega_\tau(e) = 0\,\forall e \in F',\ (\omega_\tau, \omega_{\tau\tau'})|_{F\cap E_l(\gamma)} = \xi|_{F\cap E_l(\gamma)}\big)$$

then $(\omega_\tau, \omega_{\tau\tau'})|_{F\cap E_l(\mathring{\gamma})}$ and $(\omega_\tau, \omega_{\tau\tau'})|_{F\cap E_l((\mathring{\gamma}\cup\gamma)^c)}$ are independent.

In other words: the presence of open paths in both $\omega_{\tau\tau'}$ and ω^*_τ in a fixed annuli decouples the inside from the outside of the annuli.

The index set T is then the set of sequences (F, a, b, F') as above such that the measure $\mathsf{ATRC}^{a,b}_F(\ \mid \omega_\tau(e) = 0\,\forall e \in F')$ has the l-path decoupling property. We will refer to such (F, a, b, F') as *having the l-path decoupling property.*

6.2. **Verifications of the mixing and decoupling hypotheses.** [Ott25] relies on a sequence of hypotheses, denoted Mix1, Mix2, Mar1, Mar2, Mar3. We check that these hypotheses hold in out setup.

Mixing hypotheses.

The hypotheses Mix1 and Mix2 are exponential mixing hypotheses which are weaker version of the ratio weak mixing property (Theorem 4).

Markov-type hypotheses.

Let $t = (F, a, b, F')$ be an element of our index set (volume, b.c., and constraint set with the decoupling path property).

The hypotheses Mar1, Mar2, Mar3 ask for a family of local events, $\mathrm{Ma}_i, \mathrm{Ma}^t_i$, $i \in \Gamma_l$. With our notations: local means Ma_i depends only on $E_l(\Delta_1(i))$, with $\Delta_1(i) = \{j \in \Gamma_l :\ \|j - i\|_\infty \le l\}$, and Ma^t_i depends only on $E_l(i)$. They should satisfy

(1) $\inf_i \inf_{a',b'} \mathsf{ATRC}^{a',b'}_{E_l(\Delta_1(i))}(\mathrm{Ma}_i)$ is close enough to 1,
(2) the probability of Ma^t_i is positive uniformly over i, t and the configuration outside of $E_l(i)$,
(3) if γ is a simple closed path in Γ_l, then any element of $\bigcap_{i\in\gamma:\Delta_1(i)\subset F} \mathrm{Ma}_i \cap \bigcap_{i\in\gamma:\Delta_1(i)\not\subset F, E_l(i)\cap F\neq\varnothing} \mathrm{Ma}^t_i$ implies the presence of a pair of decoupling path in $F \cap E_l(\gamma)$,
(4) if a configuration is in Ma_i, changing its value on $E_l(j)$, $j \in \Delta_1(i)$, for an element of Ma^t_j still gives a configuration in Ma_i.

The relevant local events in our setup are

- Ma_i is the event that $E_l(\Delta_1(i)) \setminus E_l(i)$ contains both an open path surrounding $E_l(i)$ in $\omega_{\tau\tau'}$, and an open dual path surrounding $E_l(i)$ in ω^*_τ,
- Ma^t_i is the event that all edges of $F \cap E_l(i)$ are open in $\omega_{\tau\tau'}$ and closed in ω_τ.

The first condition is a direct consequence of weak mixing and of exponential decay connectivities in $\omega^*_{\tau\tau}$ and in ω_τ. This requires l large enough. The second follows from finite energy. The third is the required path decoupling property of the measure P^t_Λ. The fourth is a direct consequence of our choices: Ma_i is increasing in $\omega_{\tau\tau'}$ and decreasing in ω_τ.

6.3. **Strong mixing.** We need a bit of notation to state the result we will use later. For $L \ge 1$ fixed, let $\Lambda_L(x) = x + \{-L, \dots, L\}^2$. Let $\Gamma_L = ((2L+1)\mathbb{Z})^2$. Let $X_i, Y_i, i \in \Gamma_L$ be an independent family of random variables with X_i Bernoulli R.V. of parameter p and Y_i Bernoulli R.V. of parameter q. Define $\varphi : \mathbb{Z}^2 \to \Gamma_L$ via $\varphi(x) = i$ if $x \in \Lambda_L(i)$.

For $\Lambda \subset \mathbb{Z}^2$, define

$$[\Lambda]_{\text{in}} = \{x : \ \mathrm{d}_\infty(x, \Lambda^c) > 2L\},$$
$$[\Lambda]_{\text{ext}} = \{x : \ \mathrm{d}_\infty(x, \Lambda) > 2L\}, \quad [\Lambda]_{\text{bnd}} = \mathbb{Z}^2 \setminus ([\Lambda]_{\text{in}} \cup [\Lambda]_{\text{ext}}).$$

Let $P_{\Lambda;L,p,q}$ be the law of the (site) bloc-percolation random variable ω defined as

$$\omega_x = \begin{cases} X_i & \text{if } \varphi(x) = i, \text{ and } x \in [\Lambda]_{\text{in}}, \\ Y_i & \text{if } \varphi(x) = i, \text{ and } x \in [\Lambda]_{\text{bnd}}, \\ 0 & \text{else.} \end{cases}$$

Then, the main result we obtain from [Ott25] is the next Theorem. To get it, apply the main Theorem of [Ott25] to the family P^t_Λ defined in section 6.1: the necessary hypotheses for applying the Theorem are verified in section 6.2.

Theorem 8. *Let l be as in section 6.1. For any $p > 0$, there are $L < \infty$ and $q < 1$ such that for any (F, a, b, F') having the l-path decoupling property (recall $F' \subset \partial^{\text{in}} F$), any disjoint, connected $F_1, F_2 \subset F$, and any $(f, g), (f', g') \in \{(0,0), (0,1), (1,1)\}^{F_1}$*

$$\mathrm{d}_{\text{TV}}\big(P_t\big(\cdot \mid (\xi_\tau, \xi_{\tau\tau'})|_{F_1} = (f, g)\big), P_t\big(\cdot \mid (\xi_\tau, \xi_{\tau\tau'})|_{F_1} = (f', g')\big)\big)$$
$$\leq P_{\mathbb{V}_F;L,p,q}\big([\mathbb{V}_{F_1}]_{\text{bnd}} \leftrightarrow_* [\mathbb{V}_{F_2}]_{\text{bnd}}\big)$$

where $\leftrightarrow_*$ means $*$-connections (connections with $\| \ \|_\infty$ nearest-neighbours), and

$$P_t(\cdot) = \mathsf{ATRC}^{a,b}_F\big(\cdot \ \big| \omega_\tau(e) = 0 \ \forall e \in F' \big).$$

Remark 6.1. *We will systematically apply this result in cases where $[\mathbb{V}_F]_{\text{bnd}}$ is a very elongated one dimensional object, so exponential decay of the connection probability can be obtained by a coarse-graining argument as in* [OV18, Lemma 3.2].

7. Random Walk picture in the ATRC

We start this section by introducing the objects necessary to the development of the "Ornstein-Zernike theory" (renewal picture of connection probabilities). We follow the general strategy used in [CI02, CIV03, CIV08, OV18, AOV24], most results will be imported when their proof is a repetition of existing arguments. To keep notations readable, we will use the following short-hand throughout this section:

$$\Phi \equiv \mathsf{ATRC}_{J,U}, \qquad \Phi^{a,b}_\Lambda \equiv \mathsf{ATRC}^{a,b}_{\Lambda;J,U},$$

where $\mathsf{ATRC}_{J,U}$ is the unique infinite-volume measure for the ATRC model (recall Proposition 5.9). The theorems that will be used in other sections will be stated with the notation matching the rest of the paper. Moreover, again to lighten notations, all connections are understood to be in ω_τ when not explicitly stated otherwise. We will denote $\mathcal{C}_0$ the cluster of 0 in ω_τ.

7.1. Decay rate, Cones, and Diamonds.

Start by introducing some objects.

Norm induced by the decay rates and associated sets. For $s \in \mathbb{S}^1$, define

$$\nu(s) = -\lim_{n\to\infty} \tfrac{1}{n} \ln \Phi(0 \leftrightarrow ns). \tag{26}$$

Existence of the limit follows in the standard fashion by Fekete's Lemma, using sub-additivity which follows from FKG inequality. We now extend ν to $\mathbb{R}^2$ by positive homogeneity of order one: for any $s \in \mathbb{S}^1$ and $r \geq 0$, define

$$\nu(r \cdot s) := r \cdot \nu(s).$$

Using existence of the limit and the FKG inequality, one can show that ν is a non-degenerate norm as soon as it is non-zero: ν is positive homogeneous by definition; exponential decay of connection probabilities in Φ (the ATRC model) implies positivity; the FKG inequality for Φ implies the triangular inequality for ν. See [Ale01, section 2] for details. Clearly, ν inherits the symmetries of the lattice, that is axial and diagonal reflections and rotations by $\pi/2$. From these symmetry considerations, one has that for any $s \in \mathbb{S}^1$,

$$\tfrac{1}{\sqrt{2}}\nu(\mathrm{e}_1) \leq \nu(s) \leq \sqrt{2}\nu(\mathrm{e}_1),$$

with $\mathrm{e}_1 = (1, 0)$. Moreover, one has

$$\Phi(0 \leftrightarrow x) = e^{-\nu(x)(1+o(1))}. \tag{27}$$

As ν is a norm, there are two convex sets naturally associated to it: the equi-decay set $\mathcal{U}$, and the convex set of which ν is the support function, $\mathcal{W}$.

$$\mathcal{U} = \{x \in \mathbb{R}^d : \ \nu(x) \leq 1\}, \qquad \mathcal{W} = \bigcap_{s\in\mathbb{S}^1} \{x \in \mathbb{R}^d : \ x \cdot s \leq \nu(s)\}. \tag{28}$$

It is easy to see that are dual to each other (i.e.: $\mathcal{W}$ is the equi-decay set of the norm dual to ν, and the dual norm is the support function of $\mathcal{U}$). We say that $(s, t) \in \mathbb{S}^1 \times \partial\mathcal{W}$ is a *dual pair* if $\nu(s) = t \cdot s$. From the definitions, one has, for any $x \in \mathbb{R}^d$,

$$\nu(x) = \max_{t\in\partial\mathcal{W}} t \cdot x. \tag{29}$$

We refer to [Roc70] for details on convex duality. From the last display, it is also easy to see that $\mathcal{W}$ is the closure of the convergence domain of

$$\mathbb{G}(t) = \sum_{x\in\mathbb{Z}^2} \Phi(0 \leftrightarrow x)e^{t\cdot x}. \tag{30}$$

Cones and Diamonds. Let $t \in \partial\mathcal{W}, \delta \in (0, 1)$. Let us introduce the geometric objects used in the interface study. We first define the cones and the associated diamonds:

$$\mathcal{Y}^{\blacktriangleleft}_{t,\delta} := \big\{x \in \mathbb{R}^2 : \ x \cdot t \geq (1 - \delta)\nu(x)\big\}, \quad \mathcal{Y}^{\blacktriangleright}_{t,\delta} := -\mathcal{Y}^{\blacktriangleleft}_{t,\delta},$$
$$\mathrm{Diamond}_{t,\delta}(u, v) := (u + \mathcal{Y}^{\blacktriangleleft}_{t,\delta}) \cap (v + \mathcal{Y}^{\blacktriangleright}_{t,\delta}).$$

As δ goes to 1, the cone $\mathcal{Y}^{\blacktriangleleft}_{t,\delta}$ converges to the half-plane that contains t and whose boundary is orthogonal to t. As δ goes to 0, the cone $\mathcal{Y}^{\blacktriangleleft}_{t,\delta}$ converges to the convex cone generated by the directions dual to t. The latter set is a line when ν is strictly convex.

Let $V \subset \mathbb{Z}^2$. We will say that V is:

- (t, δ)-*forward-confined* if there exists $u \in V$ such that $V \subset u + \mathcal{Y}^{\blacktriangleleft}_{t,\delta}$. When it exists, such a u is unique; we denote it by $\mathbf{f}(V)$.
- (t, δ)-*backward-confined* if there exists $v \in V$ such that $V \subset v + \mathcal{Y}^{\blacktriangleright}_{t,\delta}$. When it exists, such a v is unique; we denote it by $\mathbf{b}(V)$.
- (t, δ)-*diamond-confined* if it is both forward- and backward-confined.

We will say that $v \in V$ is a (t, δ)-*cone-point* of V if

$$V \subset v + (\mathcal{Y}^{\blacktriangleright}_{t,\delta} \cup \mathcal{Y}^{\blacktriangleleft}_{t,\delta}).$$

We denote $\mathrm{CPts}_{t,\delta}(V)$ the set of cone-points of V. When speaking about cone-points of graphs, we mean cone-points of their vertex set.

We call a graph with a distinguished vertex a *marked graph*. The distinguished vertex is denoted v^*. Define

- The sets of confined pieces (all are sets of finite connected sub-graphs of $(\mathbb{Z}^2, \mathbb{E})$):
$$\mathfrak{B}_L(t,\delta) = \{\gamma \text{ marked backward-confined with } v^* = 0\},$$
$$\mathfrak{B}_R(t,\delta) = \{\gamma \text{ marked forward-confined with } \mathbf{f}(\gamma) = 0\},$$
$$\mathfrak{A}(t,\delta) = \{\gamma \text{ diamond-confined with } \mathbf{f}(\gamma) = 0\}.$$
To fix ideas we shall, unless stated otherwise, think of $\mathfrak{A}$ as of a subset of $\mathfrak{B}_L$, that is, by default the vertex $\mathbf{f}(\gamma) = 0$ is marked for any $\gamma \in \mathfrak{A}$. Note that $\mathfrak{A}$ can alternatively be viewed as subset of $\mathfrak{B}_R$ by marking $\mathbf{b}(\gamma)$.
- The displacement along a piece:
$$\mathrm{X}(\gamma) := \begin{cases} \mathbf{b}(\gamma) & \text{if } \gamma \in \mathfrak{B}_L, \text{ in particular, if } \gamma \in \mathfrak{A}, \\ v^* & \text{if } \gamma \in \mathfrak{B}_R. \end{cases} \tag{31}$$
- The *concatenation* operation: for $\gamma_1 \in \mathfrak{B}_L$ and $\gamma_2 \in \mathfrak{B}_R$ define the concatenation of γ_2 to γ_1 as
$$\gamma_1 \circ \gamma_2 = \gamma_1 \cup (\mathrm{X}(\gamma_1) + \gamma_2).$$
The concatenation of two graphs in $\mathfrak{A}$ is an element of $\mathfrak{A}$, the concatenation of a graph in $\mathfrak{A}$ to an element of $\mathfrak{B}_L$ is an element of $\mathfrak{B}_L$, and the concatenation of a element in $\mathfrak{B}_R$ to an graph in $\mathfrak{A}$ is an element of $\mathfrak{B}_R$. The displacement along a concatenation is the sum of the displacements along the pieces.

These sets can be seen as equivalence classes of general marked forward/backward/diamond confined graphs modulo translations. A general such graph, $\gamma_L', \gamma_R', \gamma'$, can then be recovered uniquely from an element $\gamma_L, \gamma_R, \gamma$ of $\mathfrak{B}_L, \mathfrak{B}_R, \mathfrak{A}$ by specifying the translation vector.

7.2. **Main result of the section.** Our main goal is to prove the following "coupling with random walk" in infinite volume for long connections in the ATRC model. Recall that $\mathrm{e}_1 = (1,0), \mathrm{e}_2 = (0,1)$ are the canonical basis vectors.

Theorem 9. *Let $s_0 \in \mathbb{S}^1$, $t \in \partial\mathcal{W}$ dual to s_0 and $\delta \in (0,1)$ be such that the interior of $\mathcal{Y}^{\blacktriangleleft}_{t,\delta}$ contains an element of $\{\pm\mathrm{e}_1, \pm\mathrm{e}_2\}$. Then, there exist constants $\mathrm{C}, C_1, C_2 \geq 0, c_1, c_2 > 0, \epsilon > 0$ and probability measures p_L, p, p_R on $\mathfrak{B}_L(t,\delta)$, $\mathfrak{B}_R(t,\delta)$, $\mathfrak{A}(t,\delta)$ respectively such that*

(1) for any $x \in \mathbb{Z}^2$ such that $x \cdot s_0 \geq (1-\epsilon)|x|$, and any f real valued function of the cluster of 0,
$$\Big| \mathrm{C} \sum_{\gamma_L,\gamma_R} p_L(\gamma_L) p_R(\gamma_R) \sum_{k\geq 0} \sum_{\gamma_1,\dots,\gamma_k} f(\bar\gamma) \mathbb{1}_{\mathrm{X}(\bar\gamma)=x} \prod_{i=1}^{k} p(\gamma_k) - e^{t\cdot x} \mathsf{ATRC}_{J,U}\big(f(\mathcal{C}_0) \mathbb{1}_{0 \overset{\omega_\tau}{\longleftrightarrow} x}\big) \Big| \leq C_1 \|f\|_\infty e^{-c_1|x|}, \tag{32}$$
where $\bar\gamma = \gamma_L \circ \gamma_1 \circ \ldots \circ \gamma_k \circ \gamma_R$, and the sums are over $\gamma_L \in \mathfrak{B}_L$, $\gamma_R \in \mathfrak{B}_R$, and $\gamma_1, \dots, \gamma_k \in \mathfrak{A}$;

(2) for $q \in \{p_L, p_R, p\}$,
$$q\big(\mathrm{X}(\gamma) \geq \ell\big) \leq C_2 e^{-c_2 \ell}, \tag{33}$$
$$q\big(\gamma\big) \geq \alpha^{|\gamma|+1}, \tag{34}$$

for some $\alpha > 0$,

(3) there is $a > 0$ such that

$$\sum_{\gamma \in \mathfrak{A}(t,\delta)} p(\gamma)\mathrm{X}(\gamma) = a s_0. \tag{35}$$

Moreover, $\partial\mathcal{U}$, $\partial\mathcal{W}$ are analytic manifolds and the norm ν is uniformly strictly convex, that is $\mathcal{U}$ is strictly convex and $\partial\mathcal{U}$ has uniformly lower bounded curvature. In particular, each direction $s \in \mathbb{S}^1$ has a unique dual vector $t_s \in \partial\mathcal{W}$ and there exist $\kappa > 0$ such that the following sharp triangle inequality holds:

$$\nu(x) + \nu(y) - \nu(x+y) \geq \kappa(|x| + |y| - |x+y|).$$

The proof of Theorem 9 spans over the remainder of this section. It is concluded in the end of section 7.7.

Remark 7.1. *We use the value of J, U through only two inputs: the exponential decay in ω_τ, and the mixing of Theorem 4 which follows from edge relaxation (Proposition 3.2) and exponential decay in ω_τ and $\omega^*_{\tau\tau'}$. In particular, if one can extend these properties beyond the self-dual line (part of which is done in [ADG24]), Theorem 9 extends directly.*

7.3. **The coarse-graining procedure.** The first step is to analyse the typical geometry of long connections. The analysis of [CIV08, AOV24] is based on a coarse-graining of the cluster of 0. Introduce the cells: for $K, k \geq 1$ and $A \subset \mathbb{Z}^2$,

$$[A]_k = \bigcup_{x \in A} (x + \{-k, \dots, k\}^2),$$
$$\Delta_K = K\mathcal{U} \cap \mathbb{Z}^2, \quad \Delta'_K = [\Delta_K]_{\ln(K)^2}.$$

The scale parameter K will be picked large enough in the course of the proof.

As in [CIV08], we then coarse-grain the cluster of 0 using the next algorithm. Denote $\Delta \equiv \Delta_K$ and $\Delta' \equiv \Delta'_K$. For C a realization of $\mathcal{C}_0$ define $\mathrm{Sk}(C)$ via the next algorithm.

Set $v_0 = 0$, $\mathrm{Sk}_V = \{v_0\}$, $\mathrm{Sk}_E = \varnothing$, $V = \Delta'$, $i = 1$;

while $A = \big\{z \in \partial^{\mathrm{ex}} V : \ z \xleftrightarrow{(z+\Delta)\setminus V} \partial^{\mathrm{in}}(z+\Delta)\big\} \neq \varnothing$ **do**

  Set $v_i = \min A$;

  Let v^* be the smallest $v \in \mathrm{Sk}_V$ such that $v_i \in \partial^{\mathrm{in}}(\Delta + v^*)$;

  Update $\mathrm{Sk}_V = \mathrm{Sk}_V \cup \{v_i\}$, $\mathrm{Sk}_E = \mathrm{Sk}_E \cup \{\{v^*, v_i\}\}$, $V = V \cup (v_i + \Delta')$, $i = i + 1$;

end

return $(\mathrm{Sk}_V, \mathrm{Sk}_E)$;

Algorithm 1: Coarse graining of a cluster containing 0.

Denote $\mathrm{Sk}(C) = (\mathrm{Sk}_V(C), \mathrm{Sk}_E(C))$ the output of Algorithm 1 applied to $C \ni 0$. This is a tree with vertices that are elements of $\mathbb{Z}^2$. Denote by $|\mathrm{Sk}(C)|$ the number of vertices in the said tree.

7.4. **Energy-Entropy estimates.** The next step is to establish energy bounds which control the probability to see a given tree as the skeleton of the cluster.

Lemma 7.2. *For any $0 \in A \subset \mathbb{Z}^2$, and $K \geq 1$,*

$$\Phi^{1,1}_{[\Delta_K \cap A]_{\ln(K)^2}}(0 \xleftrightarrow{\Delta_K \cap A} \partial^{\mathrm{in}}\Delta_K) \leq e^{-K(1+o_K(1))}. \tag{36}$$

Where we used $o_K(1)$ for a quantity that goes to 0 as K goes to ∞. Note that the wanted probabilities are zero whenever $A \cap \Delta_K$ does not contain a path going from 0 to $\partial^{\text{in}}\Delta_K$.

Proof. By the definition of $\mathcal{U}$, the ratio weak mixing property (Theorem 4) and monotonicity:

$$\begin{aligned}\Phi^{1,1}_{[\Delta_K\cap A]_{\ln(K)^2}}(0 \xleftrightarrow{\Delta_K\cap A} \partial^{\text{in}}\Delta_K) &\leq (1 + CK^2 e^{-c'\ln(K)^2})\Phi^{0,0}_{[\Delta_K\cap A]_{\ln(K)^2}}(0 \xleftrightarrow{\Delta_K\cap A} \partial^{\text{in}}\Delta_K)\\ &\leq 2\Phi(0 \xleftrightarrow{\Delta_K} \partial^{\text{in}}\Delta_K)\\ &\leq C\sum_{x\in\partial^{\text{in}}\Delta_K} e^{-\nu(x)(1+o_K(1))} = C|\partial^{\text{in}}\Delta_K|e^{-K(1+o_K(1))}\end{aligned}$$

as soon as K is large enough. □

As a direct consequence of Lemma 7.2 and of the definition of the coarse-graining procedure, one gets the following:

Lemma 7.3 (Energy bound)**.** *There exists $K_0 \geq 0$ such that, for any $K \geq K_0$, and any $\mathcal{T} \subset \mathbb{Z}^2$,*

$$\Phi\big(\text{Sk}_V(\mathcal{C}_0) = \mathcal{T}\big) \leq e^{-K|\mathcal{T}|(1+o_K(1))},$$

where the o_K is uniform over $\mathcal{T}$

Indeed, every new vertex of the tree away from the boundary induces a connection of the form (36) in the complement of the neighbourhood of the previously explored vertices. We do not provide further details, see for example [CIV08, (2.2)] for the implementation of the bound.

Recall that $\text{Sk}(C)$ is a tree rooted at 0. Denote by Tree_N the set of all possible values of $\text{Sk}(C)$ if $|\text{Sk}(C)| = N$. We now state a general combinatorial lemma that bounds the size of Tree_N.

Lemma 7.4 (Entropy bound)**.** *There exists a universal $c > 0$ such that*

$$|\text{Tree}_\text{N}| \leq e^{c\ln(K)N}.$$

This Lemma follows from the fact that, for some $C \geq 0$, the size of Tree_N is smaller or equal to the number of N-vertex connected sub-trees of the CK^2-regular tree containing 0. The latter is bounded by $e^{c\ln(K)N}$ for some $c > 0$ by Kesten's argument [Kes82, page 85].

7.5. **Input from** [CIV08, CIV03]**: skeleton and cluster cone-points.** The main result that we import from [CIV08, CIV03] is [CIV08, Theorem 2.1] that describes a typical geometry of skeletons ([CIV08] builds on [CIV03]). The result in [CIV08] is stated for the FK-percolation, but the proof is general and relies only on Lemmata 7.3 and 7.4.

Lemma 7.5 (Skeleton Cone-points)**.** *Let $\mathcal{C}$ be a random finite connected subset of $\mathbb{Z}^2$ containing a fixed vertex v defined on some probability space $(\Omega, \mathcal{F}, P)$. Suppose that*

$$P(\text{Sk}(\mathcal{C} - v) = \mathcal{T}) \leq e^{-K|\mathcal{T}|(1+o_K(1))}.$$

Then, for any $\delta > 0$, there are $c_1, c_2 > 0$, $K_0 \geq 0$ such that, for any $K \geq K_0$, $t \in \partial\mathcal{W}$, $w \in \mathbb{Z}^2$,

$$e^{t\cdot(w-v)}P\big(w \in \mathcal{C}, |\text{CPts}_{t,\delta}(\text{Sk}(\mathcal{C} - v))| \leq c_1|w - v|/K\big) \leq e^{-c_2|w-v|}.$$

Moreover, by monotonicity, c_1, c_2, K_0 can be taken uniform over $\delta \geq \delta_0 > 0$.

The second result we import is a simple but notationally heavy use of finite energy. One can find two different implementations of this argument in [CIV08, section 2.9], and [AOV24, section 6.1]. Introduce a small variation on the notion of cone-points which will be convenient later (one could work directly with cone-points, but the equations become a bit heavier).

Recall $\mathrm{e}_1 = (1,0)$, $\mathrm{e}_2 = (0,1)$.

Definition 6 (Regular Cone-points)**.** Let $C = (V,E)$ be a connected sub-graph of $(\mathbb{Z}^2,\mathbb{E})$. Say that $v \in V$ is a *regular* (t,δ)*-cone-point* of C if it is a (t,δ)-cone-point of V and

- if $\{\pm\mathrm{e}_1\} \cap \mathcal{Y}^{\blacktriangleleft}_{t,\delta} \neq \varnothing$: $\{v,v+\mathrm{e}_1\},\{v,v-\mathrm{e}_1\} \in E$ and $\{v,v+\mathrm{e}_2\},\{v,v-\mathrm{e}_2\} \notin E$,
- if $\{\pm\mathrm{e}_2\} \cap \mathcal{Y}^{\blacktriangleleft}_{t,\delta} \neq \varnothing$ and $\{\pm\mathrm{e}_1\} \cap \mathcal{Y}^{\blacktriangleleft}_{t,\delta} = \varnothing$: $\{v,v+\mathrm{e}_2\},\{v,v-\mathrm{e}_2\} \in E$ and $\{v,v+\mathrm{e}_1\},\{v,v-\mathrm{e}_1\} \notin E$.

Denote $\mathrm{rCPts}_{t,\delta}(C)$ the set of (t,δ)-regular cone-points of C.

Note than when $\mathcal{Y}^{\blacktriangleleft}_{t,\delta}$ contains exactly one element of $\{\pm\mathrm{e}_1,\pm\mathrm{e}_2\}$, all cone-points are necessarily regular.

Lemma 7.6 (Cluster Cone-points)**.** *Let $(\Omega,\mathcal{F},P)$ be probability space, on which the following is defined. Let $v \in \mathbb{Z}^2$ be a vertex and ω be a random bond percolation configuration on $\mathbb{Z}^2$, that is a random variable with values in $\{0,1\}^{\mathbb{E}}$. The cluster of v in ω is denoted by $\mathcal{C} = \mathcal{C}(\omega)$ Suppose that*

- *ω has uniform finite energy (for opening and closing edges): for $e \in \mathbb{E}$, let $\mathcal{F}_{e^c}$ be the sigma-algebra generated by $(\omega_f)_{f\neq e}$. Then, there is $\epsilon \in (0,1)$ such that for every $e \in \mathbb{E}$,*
$$\epsilon < E(\omega_e \,|\, \mathcal{F}_{e^c}) < 1-\epsilon,$$
P-almost surely.
- *the conclusion of Lemma 7.5 holds.*

Then, for any $\delta > 0$, there are $c_1', c_2' > 0$, $L_0 \geq 1$ such that, for any $t \in \partial\mathcal{W}$ such that the interior of $\mathcal{Y}^{\blacktriangleleft}_{t,\delta}$ contains an element of $\{\pm\mathrm{e}_1,\pm\mathrm{e}_2\}$, and any $w \in \mathbb{Z}^2 \cap \mathcal{Y}^{\blacktriangleleft}_{t,\delta}$ with $|w-v| \geq L_0$,
$$e^{t\cdot(w-v)}P\big(w \in \mathcal{C},\ |\mathrm{rCPts}_{t,\delta}(\mathcal{C})| \leq c_1'|w-v|\big) \leq e^{-c_2'|v-w|}.$$
By monotonicity, c_1', c_2', L_0 can be taken uniform over $\delta \geq \delta_0 > 0$.

The idea for going from Lemma 7.5 to Lemma 7.6 is simple: when $v \leftrightarrow w$, up to an exponentially small error, there must be at least $c|v-w|/K$ cone-points of the skeleton by Lemma 7.5. Up to anther exponentially small error, a positive fraction of these cone-points must then be regular cone-points of $\mathcal{C}$ by uniform finite energy.

Remark 7.7. *Note that both Lemmas hold directly when w is not in $v + \mathcal{Y}^{\blacktriangleleft}_{t,\delta}$: then $P(w \in \mathcal{C}) \leq e^{-\nu(w-v)(1+o(1))}$ but $t\cdot(w-v) - \nu(w-v) \leq -\delta\nu(w-v)$ by definition of $\mathcal{Y}^{\blacktriangleleft}_{t,\delta}$.*

7.6. **Cone-points and pre-renewal structure.** The first needed result is the presence of many regular cone-points.

Theorem 10. *Let $\delta \in (0,1)$. There exist $C \geq 0, c_1, c_2 > 0$ such that for any $t \in \partial\mathcal{W}$ such that the interior of $\mathcal{Y}^{\blacktriangleleft}_{t,\delta}$ contains an element of $\{\pm\mathrm{e}_1,\pm\mathrm{e}_2\}$, and any $x \in \mathbb{Z}^2$,*
$$e^{t\cdot x}\Phi\big(0 \leftrightarrow x, |\mathrm{rCPts}_{t,\delta}(\mathcal{C}_0)| \leq c_1|x|\big) \leq Ce^{-c_2|x|}.$$

Proof. The result follows from Lemmas 7.5, and 7.6: Lemma 7.3 provides the required bound on the probability of a given skeleton, and the model has finite energy, which are the needed hypotheses for the Lemmas. As in these Lemmas, the claim is trivial when $x \notin \mathcal{Y}^{\blacktriangleleft}_{t,\delta}$. □

We now fix $t_0 \in \partial\mathcal{W}$, $\delta \in (0,1)$ such that the interior of $\mathcal{Y}^{\blacktriangleleft}_{t_0,\delta}$ contains and element of $\{\pm \mathrm{e}_1, \pm \mathrm{e}_2\}$, and we set

$$\begin{aligned} \mathrm{rCPts} \equiv \mathrm{rCPts}_{t_0,\delta}, \quad \mathcal{Y}^{\blacktriangleleft} \equiv \mathcal{Y}^{\blacktriangleleft}_{t_0,\delta}, \quad \mathcal{Y}^{\blacktriangleright} \equiv \mathcal{Y}^{\blacktriangleright}_{t_0,\delta}, \\ \mathfrak{B}_L \equiv \mathfrak{B}_L(t_0,\delta), \quad \mathfrak{B}_R \equiv \mathfrak{B}_R(t_0,\delta), \quad \mathfrak{A} \equiv \mathfrak{A}(t_0,\delta). \end{aligned} \tag{37}$$

The idea is now to write $\mathcal{C}_0$ as a concatenation of "irreducible graphs" by splitting it at its regular cone-points. This will lead to a structure that, graphically, looks like a renewal structure. The last step will then be to extract a *real* renewal structure (at the level of the measure) from this graphical one. To this end, introduce the notion of *irreducible graphs*. Say that

- A marked backward-confined graph (γ_L, v^*) is *reducible* if the diamond $\mathrm{Diamond}(v^*, \mathbf{b}(\gamma_L))$ contains a regular cone-point of γ_L and *irreducible* if not. I.e.: (γ_L, v^*) is irreducible if
$$\mathrm{Diamond}(v^*, \mathbf{b}(\gamma_L)) \cap \mathrm{rCPts}(\gamma_L) = \varnothing.$$
- A marked forward-confined graph (γ_R, v^*) is *reducible* if the diamond $\mathrm{Diamond}(\mathbf{f}(\gamma_R), v^*)$ contains a regular cone-point of γ_R and *irreducible* if not. I.e.: (γ_R, v^*) is irreducible if
$$\mathrm{Diamond}(\mathbf{f}(\gamma_R), v^*) \cap \mathrm{rCPts}(\gamma_R) = \varnothing.$$
- A diamond-confined graph γ is *reducible* if it contains a regular cone-point, and *irreducible* if not. I.e.: γ is irreducible if
$$\mathrm{rCPts}(\gamma) = \varnothing.$$

In words: irreducible graphs are those that cannot be written as the concatenation of two non-trivial graphs, so that the concatenation point is a regular cone-point. We will denote the sets of irreducible marked graphs in, respectively, $\mathfrak{B}_L$, $\mathfrak{B}_R$, $\mathfrak{A}$ by

$$\mathfrak{B}_L^{\mathrm{ir}}, \quad \mathfrak{B}_R^{\mathrm{ir}}, \quad \mathfrak{A}^{\mathrm{ir}}.$$

For $x \in \mathcal{Y}^{\blacktriangleleft} \cap \mathbb{Z}^2$, and a connected component $C_0 \ni 0, x$ containing at least two regular cone points v_1, v_2 with $0 \in v_i + \mathcal{Y}^{\blacktriangleright}$, $x \in v_i + \mathcal{Y}^{\blacktriangleleft}$, we can introduce the splitting into irreducible components:

$$C_0 = \eta_L \sqcup \eta_1 \sqcup \cdots \sqcup \eta_M \sqcup \eta_R,$$

where $M \geq 1$, $(\eta_L, 0)$, (η_R, x), $\eta_1, \dots, \eta_M$ are all irreducible, confined, (marked) graphs, and $\sqcup$ means disjoint union of edges (there are sites overlap at cone-points). Now, as mentioned in the end of section 7.1, there is a bijection between pairs $(\tilde{\gamma}, v) \in \mathfrak{A} \times \mathbb{Z}^2$ and diamond-confined connected graphs (translate $\tilde{\gamma}$ by v to obtained the graph $\gamma = v + \tilde{\gamma}$). Similar considerations hold for marked forward/backward confined connected graphs. In particular, for η_i in the above decomposition, there is a unique $w_i \in \mathbb{Z}^2$ and a unique $\tilde{\eta}_i \in \mathfrak{A}^{\mathrm{ir}}$ such that $\eta_i = w_i + \tilde{\eta}_i$. Similarly for η_L, η_R. As the marked point of η_L is 0, one has directly

$$w_L = 0, \quad w_1 = \mathrm{X}(\tilde{\eta}_L), \quad w_2 = \mathrm{X}(\tilde{\eta}_L \circ \tilde{\eta}_1), \dots \quad w_R = \mathrm{X}(\tilde{\eta}_L \circ \tilde{\eta}_1 \circ \dots \circ \tilde{\eta}_M),$$

and the equivalent writing of C_0:

$$\eta_L \sqcup \eta_1 \sqcup \cdots \sqcup \eta_M \sqcup \eta_R = \tilde{\eta}_L \circ \tilde{\eta}_1 \circ \dots \circ \tilde{\eta}_M \circ \tilde{\eta}_R, \qquad \mathrm{X}(\tilde{\eta}_L \circ \tilde{\eta}_1 \circ \dots \circ \tilde{\eta}_R) = x,$$

with $\tilde{\eta}_1,\ldots,\tilde{\eta}_M \in \mathfrak{A}^{\mathrm{ir}}$, $\tilde{\eta}_L \in \mathfrak{B}^{\mathrm{ir}}_L$, and $\tilde{\eta}_R \in \mathfrak{B}^{\mathrm{ir}}_R$.

By our definition of "regular cone points", for any $\tilde{\eta} \in \mathfrak{A}^{\mathrm{ir}}$, there is only one edge in $\tilde{\eta}$ containing $\mathbf{f}(\tilde{\eta})$, and the same for $\mathbf{b}(\tilde{\eta})$. Denote these edges $\mathbf{f}_e(\tilde{\eta}), \mathbf{b}_e(\tilde{\eta})$ (and define similarly $\mathbf{b}_e(\tilde{\eta}_L)$ and $\mathbf{f}_e(\tilde{\eta}_R)$).

For $\tilde{\eta}, \tilde{\eta}_L, \tilde{\eta}_R$ as above, and $v \in \mathbb{Z}^2$, introduce the percolation events $A_v(\tilde{\eta})$ to be the event that

- the edges in $(v + \tilde{\eta} \setminus \{\mathbf{f}_e(\tilde{\eta}), \mathbf{b}_e(\tilde{\eta})\})$ are open in ω_τ,
- $v + \mathbf{f}_e(\tilde{\eta})$, and $v + \mathbf{b}_e(\tilde{\eta})$ are open in $\omega_{\tau\tau'}$ and closed in ω_τ,
- $\partial^{\mathrm{ex}}(v + \tilde{\eta})$ are closed in ω_τ.

Define similarly $A_v(\tilde{\eta}_L)$, $A_v(\tilde{\eta}_R)$.

Now, swapping an edge e which links two clusters in ω_τ from open to close generate a weight $\frac{2^1 2^1}{2^2 2^0} = 1$ as edges have weight two (2^1 in the numerator), as do have clusters (2^1 in the numerator), and closing the edge (2^0 in the denominator) makes the configuration pass from one to two distinct clusters (2^2 in the denominator). Every edge of the form $\mathbf{b}_e, \mathbf{f}_e$ always separates two distinct clusters of ω_τ, so one has

$$\Phi(\mathcal{C}_0 = \eta_L \sqcup \eta_1 \sqcup \cdots \sqcup \eta_M \sqcup \eta_R) = \Phi\big(A_{w_L}(\tilde{\eta}_L), A_{w_1}(\tilde{\eta}_1), \ldots, A_{w_R}(\tilde{\eta}_R)\big).$$

Slightly abusing notation, we will write

$$\tilde{\eta}_L \circ \tilde{\eta}_1 \circ \ldots \circ \tilde{\eta}_M \circ \tilde{\eta}_R \equiv A_{w_L}(\tilde{\eta}_L) \cap A_{w_1}(\tilde{\eta}_1) \cap \cdots \cap A_{w_R}(\tilde{\eta}_R),$$

and

$$\Phi(\tilde{\eta} \,|\, \tilde{\eta}_L \circ \tilde{\eta}_1 \circ \ldots \circ \tilde{\eta}_k) \equiv \Phi\big(A_{\mathrm{X}(\bar{\eta})}(\tilde{\eta}) \,|\, \tilde{\eta}_L \circ \tilde{\eta}_1 \circ \ldots \circ \tilde{\eta}_k\big)$$

where $\bar{\eta} = \tilde{\eta}_L \circ \tilde{\eta}_1 \circ \ldots \circ \tilde{\eta}_k$.

This leads to the following conditional probability decomposition:

$$\Phi(\tilde{\eta}_L \circ \tilde{\eta}_1 \circ \ldots \circ \tilde{\eta}_M \circ \tilde{\eta}_R) = \Phi(\tilde{\eta}_L)\Phi(\tilde{\eta}_1 \,|\, \tilde{\eta}_L)\Phi(\tilde{\eta}_2 \,|\, \tilde{\eta}_L \circ \tilde{\eta}_1) \ldots . \tag{38}$$

Let us describe informally the last step: one has represented the probability of a given cluster participating to the event $0 \leftrightarrow x$ as a product of *dependent* kernels (which are not probability kernels). The idea is now to represent this product of dependent kernels as an alternative product of *independent* (factorized) kernels defined on diamond-confined graphs, which will then be suitably normalize by a factor $e^{t_0 \cdot \mathrm{X}(\gamma)}$ to obtain probability kernels. We will strongly borrow from [AOV24], so we only present the necessary modifications and refer to the relevant section once everything is in place.

7.7. **Mixing for weights and renewal structure.** We will use the same procedure as [AOV24, section 7] with the tricks from [OV18, Appendix C] to compensate for the lake of monotonicity in the conditional kernels. We only describe the needed inputs, and will refer to [AOV24, section 7] once we arrive at a stage where the remaining arguments is a copy-pasting of [AOV24, section 7].

We start by proving a mixing result for the conditional probabilities.

Lemma 7.8. *Let $\delta \in (0,1), t_0 \in \partial\mathcal{W}$ be the values fixed in (37), and use the notations of (37). Then, there exist $\rho > 0, C \geq 0, c > 0$, such that*

- *for every $\gamma \in \mathfrak{B}^{\mathrm{ir}}_R$, one has that*

$$\inf_{n\geq 0} \inf_{\tilde{\eta}_0 \in \mathfrak{B}^{\mathrm{ir}}_L} \inf_{\tilde{\eta}_1,\ldots,\tilde{\eta}_n \in \mathfrak{A}^{\mathrm{ir}}} \Phi(\tilde{\eta} \,|\, \tilde{\eta}_0 \circ \ldots \circ \tilde{\eta}_n) \geq \rho^{|\tilde{\eta}|}; \tag{39}$$

- *for any* $n, k, k' \geq 0$, *any* $\tilde{\eta}_0, \tilde{\eta}'_0 \in \mathfrak{B}^{\text{ir}}_L$, *any* $\tilde{\eta} \in \mathfrak{B}^{\text{ir}}_R \cup \mathfrak{A}^{\text{ir}}$, *and any* $\tilde{\eta}_1, \dots, \tilde{\eta}_{k+n}, \tilde{\eta}'_1, \dots, \tilde{\eta}_{k'} \in \mathfrak{A}^{\text{ir}}$

$$\Big|\frac{\Phi\big(\tilde{\eta}\,\big|\,\tilde{\eta}_0 \circ \tilde{\eta}_1 \dots \circ \tilde{\eta}_k \circ \tilde{\eta}_{k+1} \dots \circ \tilde{\eta}_{k+n}\big)}{\Phi\big(\tilde{\eta}\,\big|\,\tilde{\eta}'_0 \circ \tilde{\eta}'_1 \dots \circ \tilde{\eta}'_{k'} \circ \tilde{\eta}_{k+1} \dots \circ \tilde{\eta}_{k+n}\big)} - 1\Big| \leq Ce^{-cn}. \tag{40}$$

Proof. The first point is finite energy. Focus on the second. Let $\tilde{\eta}'_{k'+i} \equiv \tilde{\eta}_{k+i}$ for $i = 1, \dots n$. For $l \geq 1$, let

$$v_l = \mathrm{X}(\tilde{\eta}_l), \quad v'_l = \mathrm{X}(\tilde{\eta}'_l), \quad w_l = \mathrm{X}(\tilde{\eta}_0 \circ \tilde{\eta}_1 \dots \circ \tilde{\eta}_l), \quad w'_l = \mathrm{X}(\tilde{\eta}'_0 \circ \tilde{\eta}_1 \dots \circ \tilde{\eta}'_l).$$

Let also

$$u_l = w_{k+l} - w_k = \mathrm{X}(\tilde{\eta}_{k+1} \circ \dots \circ \tilde{\eta}_{k+l}) = w'_{k'+l} - w'_{k'}.$$

For $v \in \mathbb{Z}^2$, introduce the (translations of) the events corresponding to a given chain of irreducible graphs:

$$B_{L,v} = A_v(\tilde{\eta}_0) \cap \bigcap_{i=1}^{k} A_{v+w_{i-1}}(\tilde{\eta}_i), \quad B'_{L,v} = A_v(\tilde{\eta}'_0) \cap \bigcap_{i=1}^{k'} A_{v+w'_{i-1}}(\tilde{\eta}'_i),$$

$$B_v = \bigcap_{l=1}^{n} A_{v+u_l}(\tilde{\eta}_{k+l}), \quad B_{R,v} = A_{v+u_n}(\tilde{\eta}).$$

Now, using translation invariance of Φ,

$$\begin{aligned}\frac{\Phi\big(\tilde{\eta}\,\big|\,\tilde{\eta}_0 \circ \dots \circ \tilde{\eta}_{k+n}\big)}{\Phi\big(\tilde{\eta}\,\big|\,\tilde{\eta}'_0 \circ \dots \circ \tilde{\eta}'_{k'+n}\big)} &= \frac{\Phi\big(B_{R,w_k}\,\big|\,B_{L,0} \cap B_{w_k}\big)}{\Phi\big(B_{R,w'_k}\,\big|\,B'_{L,0} \cap B_{w'_{k'}}\big)} = \frac{\Phi\big(B_{R,0}\,\big|\,B_{L,-w_k} \cap B_0\big)}{\Phi\big(B_{R,0}\,\big|\,B'_{L,-w'_{k'}} \cap B_0\big)} \\ &= \frac{\Phi\big(B_{R,0} \cap B_{L,-w_k}\,\big|\,B_0\big)}{\Phi\big(B_{R,0}\,\big|\,B_0\big)\Phi\big(B_{L,-w_k}\,\big|\,B_0\big)} \frac{\Phi\big(B_{R,0}\,\big|\,B_0\big)\Phi\big(B'_{L,-w'_{k'}}\,\big|\,B_0\big)}{\Phi\big(B'_{L,-w'_{k'}} \cap B_{R,0}\,\big|\,B_0\big)}. \end{aligned} \tag{41}$$

Now, $B_{R,0}$ is supported on the edges with at least one endpoint in $u_n + \mathcal{Y}^{\blacktriangleleft}$, denoted $E_{\triangleleft}(u_n)$, whilst $B_{L,-w_k}, B'_{L,-w'_{k'}}$ are supported on the edges with at least one endpoint in $-\mathcal{Y}^{\blacktriangleleft}$, denoted $E_{\triangleright}$. The claim will follow from suitable exponential ratio mixing of $P(\cdot) := \Phi(\cdot\,|\,B_0)$. Divide the proof of ratio mixing into two claims. We start by proving mixing of P. We will use the following observation: let γ be the simple closed dual path surrounding the connected graph $\tilde{\eta}_1 \circ \dots \circ \tilde{\eta}_n$. Let $*\gamma$ be the set of primal edges that are crossed by γ, and denote V_γ the set of sites surrounded by γ. Then,

$$P|_{\mathbb{E}\setminus\mathbb{E}_{V_\gamma}} = \mathsf{ATRC}^{1,1}_{\mathbb{E}\setminus\mathbb{E}_{V_\gamma}}\big(\ \big|\,\omega_\tau|_{*\gamma} = 0\big) = \mathsf{ATRC}^{0,1}_{\mathbb{E}\setminus\mathbb{E}_{V_\gamma}}\big(\ \big|\,\omega_\tau|_{*\gamma} = 0\big), \tag{42}$$

where $\omega_\tau|_{*\gamma}$ is the restriction of ω_τ to $*\gamma$.

Claim 1. *There exist* $C \geq 0, c > 0$ *such that, for any* $n \geq 0$, *any* $\tilde{\eta}_1, \dots, \tilde{\eta}_n \in \mathfrak{A}^{\text{ir}}$, *any finite sets of edges* $F_1 \subset E_{\triangleright}$, $F_2 \subset E_{\triangleleft}$, *any* F_1*-measurable event* A *and any* F_2*-measurable event* B,

$$|P(A, B) - P(A)P(B)| \leq CP(A) \sum_{e \in F_1} \sum_{f \in F_2} e^{-c\mathrm{d}(e,f)},$$

where P *is defined as above the claim.*

Proof. Looking at (42), $P|_{\mathbb{E}\setminus\mathbb{E}_{V_\gamma}}$ has the l-path decoupling property of section 6.1. One can therefore use Theorem 8 to obtain the claim (see remark 6.1 after Theorem 8 for how to get from the statement of Theorem 8 to the present exponential decay). □

We then turn this into ratio mixing using Appendix A.1.

Claim 2. *There exists $C \geq 0, c > 0$ such that for any $n \geq 0$, any $v \in \mathbb{Z}^2$, any $\tilde{\eta}_1, \dots, \tilde{\eta}_n \in \mathfrak{A}^{\mathrm{ir}}$ with $\mathrm{X}(\tilde{\eta}_1 \circ \dots \circ \tilde{\eta}_n) = v$, any $F_\triangleright \subset E_\triangleright$, $F_\triangleleft \subset E_\triangleleft(v)$ finite, and any $\alpha \in \{(0,0),(0,1),(1,1)\}^{F_\triangleleft}$,*

$$\sup_{\xi,\xi'} \Big| \frac{P(X_{F_\triangleleft} = \alpha \,|\, X_{F_\triangleright} = \xi)}{P(X_{F_\triangleleft} = \alpha \,|\, X_{F_\triangleright} = \xi')} - 1 \Big| \leq C e^{-c|v|},$$

where the sup is over ξ, ξ' having positive probability, P is as defined before Claim 1, and $X_F = (\omega_\tau|_F, \omega_{\tau\tau'}|_F)$.

Proof. The proof will be an application of Lemma A.1. Recall (42). We first prove the claim for $|v|$ large (larger than some number depending on the angular aperture of $\mathcal{Y}^\blacktriangleleft$). We also implicitly work in a large finite volume, $\Lambda_N = \{-N, \dots, N\}^2$, with $0, 1$ boundary conditions and take limits afterwards (everything being uniform over the volume).

Let $\epsilon_0 > 0$, and let Δ be the set of edges with an endpoint in $\Lambda_N \setminus (\frac{2}{3}v + \mathcal{Y}^\blacktriangleleft_{t_0,\delta+\epsilon_0} \cup \frac{1}{3}v + \mathcal{Y}^\blacktriangleright_{t_0,\delta+\epsilon_0})$. We apply Lemma A.1 with

- $\Omega_1 = (\{0,1\}^2)^{F_\triangleleft}$, $\Omega_2 = (\{0,1\}^2)^{\Delta}$,
- $\mu = \Phi(\;|\,\eta_1 \circ \dots \circ \eta_n, X_{F_\triangleright} = \xi)|_{F_\triangleleft \sqcup \Delta}$, $\nu = \Phi(\;|\,\eta_1 \circ \dots \circ \eta_n, X_{F_\triangleright} = \xi')|_{F_\triangleleft \sqcup \Delta}$,
- $D \subset \Omega_2$ is the event that there is an open path in $\omega_{\tau\tau'}$ from the top boundary of Λ_N to its bottom boundary staying in Δ, and that there is a dual path of open edges in ω^*_τ from the top boundary of Λ_N to its bottom boundary using only edges dual to edges in Δ.

The hypotheses of Lemma A.1 hold with $\epsilon = e^{-c|v|}$ (when $|v|$ is large enough) by the strong mixing property of $\Phi(\;|\,\eta_1 \circ \dots \circ \eta_n)$, see Claim 1, and the uniform exponential decay of $\omega^*_{\tau\tau'}, \omega_\tau$.

Remains to deal with $\|v\|_\infty$ less than some constant K. Let $M \geq K$ be some fixed large number. Then, for any $\tilde{\eta}_1, \dots, \tilde{\eta}_n$ with $\mathrm{X}(\tilde{\eta}_1 \circ \dots \circ \tilde{\eta}_n) = v$, and $\xi, \xi' \in \{(0,0),(0,1),(1,1)\}^{F_\triangleright}, \alpha \in \{(0,0),(0,1),(1,1)\}^{F_\triangleleft}$, one can find events A_L, A'_L, B_L, B'_L, and A_0, A_R, such that

$$\begin{aligned}
\{X_{F_\triangleright} = \xi\} \cap \{X_{F_\triangleleft} = \alpha\} \cap \tilde{\eta}_1 \circ \dots \circ \tilde{\eta}_n &= A_L \cap B_L \cap A_0 \cap A_R,\\
\{X_{F_\triangleright} = \xi'\} \cap \{X_{F_\triangleleft} = \alpha\} \cap \tilde{\eta}_1 \circ \dots \circ \tilde{\eta}_n &= A'_L \cap B'_L \cap A_0 \cap A_R,\\
\{X_{F_\triangleright} = \xi\} \cap \tilde{\eta}_1 \circ \dots \circ \tilde{\eta}_n &= A_L \cap B_L \cap A_0,\\
\{X_{F_\triangleright} = \xi'\} \cap \tilde{\eta}_1 \circ \dots \circ \tilde{\eta}_n &= A'_L \cap B'_L \cap A_0,
\end{aligned}$$

and with A_L, A'_L supported on $F_\triangleright \setminus \mathbb{E}_{\Lambda_M}$, A_R supported on $F_\triangleleft \setminus \mathbb{E}_{\Lambda_M}$, and A_0, B_L, B'_L supported on $\mathbb{E}_{\Lambda_M}$ (here, we used $\tilde{\eta}_1 \circ \dots \circ \tilde{\eta}_n$ to mean the associated percolation event). Then, by the ratio weak mixing property of Φ (Theorem 4) and finite energy (for M large enough),

$$2^{-1} \leq \frac{\Phi(A_L, A_R)}{\Phi(A_L)\Phi(A_R)} \leq 2, \quad 2^{-1} \leq \frac{\Phi(A_L, A'_R)}{\Phi(A_L)\Phi(A'_R)} \leq 2,$$
$$\Phi(B_L, A_0 \,|\, A_L), \Phi(B'_L, A_0 \,|\, A'_L, A_R) \geq e^{-cM^2},$$

with $c > 0$. Putting these together,

$$\begin{aligned}
\frac{\Phi(X_{F_\triangleleft} = \alpha \,|\, \tilde{\eta}_1 \circ \dots \circ \tilde{\eta}_n, X_{F_\triangleright} = \xi)}{\Phi(X_{F_\triangleleft} = \alpha \,|\, \tilde{\eta}_1 \circ \dots \circ \tilde{\eta}_n, X_{F_\triangleright} = \xi')} &= \frac{\Phi(A_L, B_L, A_0, A_R)\Phi(A'_L, B'_L, A_0)}{\Phi(A_L, B_L, A_0)\Phi(A'_L, B'_L, A_0, A_R)}\\
&\leq \frac{\Phi(A_L, A_R)\Phi(A'_L)}{\Phi(A_L)\Phi(B_L, A_0 \,|\, A_L)\Phi(A'_L, A_R)\Phi(B'_L, A_0 \,|\, A'_L, A_R)} \leq 4e^{2cM^2},
\end{aligned}$$

which concludes the proof. □

We can now apply Claim 2 to (41) to obtain the result by noting that the displacement of $\tilde{\eta}_1 \circ \ldots \circ \tilde{\eta}_n$ is, at least, a strictly positive multiple of n. □

For $k > l \geq 1$, $\tilde{\eta}_0 \in \mathfrak{B}_L^{\mathrm{ir}}$, $\tilde{\eta}_1, \ldots, \tilde{\eta}_{k-1} \in \mathfrak{A}^{\mathrm{ir}}$, and $\tilde{\eta} \in \mathfrak{A}^{\mathrm{ir}} \cup \mathfrak{B}_R^{\mathrm{ir}}$, set

$$a_0(\tilde{\eta} \,|\, \tilde{\eta}_0, \ldots, \tilde{\eta}_{k-1}) \equiv a_0(\tilde{\eta}) = \inf_{n\geq 0} \inf_{\zeta_0 \in \mathfrak{B}_L^{\mathrm{ir}}} \inf_{\zeta_1,\ldots,\zeta_n \in \mathfrak{A}^{\mathrm{ir}}} \Phi\big(\tilde{\eta} \,\big|\, \zeta_0 \circ \zeta_1 \circ \ldots \circ \zeta_n\big),$$

$$a_k(\tilde{\eta} \,|\, \tilde{\eta}_0, \ldots, \tilde{\eta}_{k-1}) = \Phi(\tilde{\eta} \,|\, \tilde{\eta}_0 \circ \ldots \circ \tilde{\eta}_{k-1}),$$

and

$$a_l(\tilde{\eta} \,|\, \tilde{\eta}_0, \ldots, \tilde{\eta}_{k-1}) \equiv a_l(\tilde{\eta} \,|\, \tilde{\eta}_{k-l}, \ldots, \tilde{\eta}_{k-1}) = \inf_{n\geq 0} \inf_{\zeta_0 \in \mathfrak{B}_L^{\mathrm{ir}}} \inf_{\zeta_1,\ldots,\zeta_n \in \mathfrak{A}^{\mathrm{ir}}} \Phi\big(\tilde{\eta} \,\big|\, \zeta_0 \circ \ldots \circ \zeta_n \circ \tilde{\eta}_{k-l} \circ \ldots \circ \tilde{\eta}_{k-1}\big)$$

In worlds: a_k records the minimal mass given by a conditional probability to an irreducible graph for a fixed frozen "recent" past, and any "less recent" possible past. Introduce then the mass increments (p_k has the same definition domain as a_k)

$$p_0 = a_0, \quad p_k = a_k - a_{k-1}.$$

Note that by definition of the a_k's, the p_k's are always non-negative. This property is what compensates the lake of monotonicity of $\Phi(\tilde{\eta} \,|\, \tilde{\eta}_0 \circ \ldots \circ \tilde{\eta}_{k-1})$, which is the quantity corresponding to $q(\gamma_k \,|\, \gamma_0, \ldots, \gamma_{k-1})$ in [AOV24] (monotonicity is property P6 there). One can now duplicate [AOV24, section 7] to conclude the proof of Theorem 9 with the following adaptations: use the p_k's defined above in place of the ones defined in [AOV24], and use Lemma 7.8 as a replacement for [AOV24, properties P5, P7].

7.8. **OZ asymptotics.** As a first application of Theorem 9, one obtains Ornstein-Zernike asymptotics for the two-point function of the AT model (Theorem 3).

Proof of Theorem 3. Theorem 9 gives, in particular, that for $s \in \mathbb{S}^1$, one can find $t \in \partial\mathcal{W}, \delta \in (0,1)$ such that (sums are over pieces confined in the cones/diamonds obtained using $\mathcal{Y}_{t,\delta}^{\blacktriangleleft}$)

$$\Big| \mathrm{C} \sum_{\gamma_L,\gamma_R} p_L(\gamma_L) p_R(\gamma_R) \sum_{k\geq 0} \sum_{\gamma_1,\ldots,\gamma_k} \mathbb{1}_{\mathrm{X}(\bar{\gamma})=ns} \prod_{i=1}^k p(\gamma_k) - e^{n\nu(s)} \mathsf{ATRC}_{J,U}(0 \xleftrightarrow{\omega_\tau} ns) \Big| \leq C_1 e^{-c_1 n},$$

which is a comparison between $e^{n\nu(s)}\mathsf{ATRC}_{J,U}(0 \leftrightarrow ns)$ and the Green functions of a directed random walk on $\mathbb{Z}^2$ (the push-forward of p_L, p_R, p by X). One can then use the local limit theorem in dimension 2 as in [AOV24, section 8.1] to obtain

$$e^{n\nu(s)}\mathsf{ATRC}_{J,U}(0 \leftrightarrow ns) = \frac{c(s)}{\sqrt{n}}(1 + o_n(1)), \tag{43}$$

which is the wanted asymptotics for the ATRC model. The claim for the AT model follows from the coupling (5) between AT and ATRC, . □

8. Invariance principle

8.1. **Notations and main result of the section.** As in section 7, we will describe the geometry of the cluster of v_L under $\mathsf{mATRC}_{n,m}(\ |\, v_L \leftrightarrow v_R)$ (defined in section 4) using a coupling with a random walk bridge in such a way that the cluster is included in diamonds with endpoints at the random walk steps. We will focus on direction e_1 for simplicity, but the analysis can easily be adapted to other directions at the cost of slightly heavier notations.

Recall the setup of section 4: let $\Lambda = \Lambda_{n,m} = \{-n, \dots, n\} \times \{-m, \dots, m\}$,

$$E = \mathbb{E}_\Lambda \cup \{e \in \partial^{\text{edge}}\mathbb{E}_\Lambda :\ e \subset \mathbb{H}_+\},$$
$$V = V_{n,m} = \mathbb{V}_E, \quad E_b^+ = \big\{\{i,j\} \in \mathbb{E} :\ i,j \notin \Lambda, i,j \in V\big\}.$$

Recall also

$$\partial_-\Lambda = \{i \in \partial^{\text{in}}\Lambda :\ i \in \mathbb{H}_-\}.$$

As in the previous section, $\Phi \equiv \mathsf{ATRC}$. We will denote (for notational convenience, we will only stress the m dependence, as the n one is obvious)

$$\Phi_m \equiv \Phi_{n,m} = \mathsf{mATRC}_{n,m}|_{E\cup E_b^+}$$

the measure $\mathsf{mATRC}_{n,m}$ restricted to $E \cup E_b^+$. Φ_m is supported on $\Omega_m = \{(0,0),(0,1),(1,1)\}^E \times \{(0,0),(1,1)\}^{E_b^+}$. Denote also for $F \subset E \cup E_b^+$, and $(a,b) \in \Omega_m$,

$$\Phi^{a,b}_{m;F} := \Phi_m\big(\ |\, \xi_\tau(e) = a_e, \xi_{\tau\tau'}(e) = b_e\, \forall e \in F^c\big)|_F \tag{44}$$

its conditional version. Recall its main properties:

(1) The probability of a given pair $a, b \in \Omega_m$ is given by

$$\Phi_m(a,b) \propto \prod_{e\in E} 2^{a_e}(\mathbf{c}-2)^{b_e-a_e} \prod_{e\in E_\mathrm{b}^+}\Big(\frac{2}{\mathbf{c}_b - 1}\Big)^{a_e} \cdot 2^{\kappa_+(a)} \prod_{C\in\mathrm{cl}(b)}\big(\mathbb{1}_{C\subset\Lambda} + \mathbf{c}_b^{\sum_{i\in\partial_-\Lambda\cap C} d_i}\big)$$

where:
- $d_i = 1$ for all i in $\partial_-\Lambda$ but the lower corners, and equals 2 at the corners;
- $\kappa_+(a)$ is the number of connected components in the graph $(\mathbb{V}_{E\cup E_b^+}, a)$;
- $\mathrm{cl}(b)$ is the set of connected components in the graph $(\mathbb{V}_{E\cup E_b^+}, b)$.

(2) Φ_m is strong-FKG (satisfies the FKG-lattice condition).

(3) $\Phi^{a,b}_{m;E} \preccurlyeq \mathsf{ATRC}^{1,1}_E$.

In particular, for any $F \subset \mathbb{E}_{\Lambda_{n-1,m-1}}$, $F' \subset E$, and any $(a,b) \in \Omega_m$ one has

$$\mathsf{ATRC}^{0,0}_F \preccurlyeq \Phi^{a,b}_{m;F}, \quad \Phi^{a,b}_{m;F'} \preccurlyeq \mathsf{ATRC}^{1,1}_{F'}. \tag{45}$$

For the lower comparison: use the monotonicity for Φ_m to closed edges in F^c and note that when all edges of F^c are closed, the measure on F is $\mathsf{ATRC}^{0,0}_F$. The upper comparison follows from strong-FKG and $\Phi_m \preccurlyeq \mathsf{ATRC}^{1,1}_E$.

Recall that $v_L = (-n-1, 0)$ and $v_R = (n+1, 0)$, and that we wish to study the cluster of v_L under $\Phi_m\big(\ |\, v_L \overset{\xi_\tau}{\longleftrightarrow} v_R\big)$. For the sake of readability, all connections will be understood to take place in ξ_τ if not specified otherwise. We denote $(\xi_\tau, \xi_{\tau\tau'})$ a sample from Φ_m, and $\mathcal{C}$ the cluster of v_L in the ξ_τ marginal.

We will use the probabilistic description of long clusters from Theorem 9 with a suitable choice of cones. First, note that by symmetry the (unique by Theorem 9) $t \in \partial\mathcal{W}$ dual to e_1 is $t = \nu_1\mathrm{e}_1$. Then, by strict convexity and symmetry, the cones $\mathcal{Y}^{\blacktriangleleft}_{\nu_1\mathrm{e}_1,\delta}$ are invariant under reflection through $\{x :\ x_2 = 0\}$, and have an angular aperture continuous in δ, strictly increasing, which converges to 0 as $\delta \to 0$ and to π as $\delta \to 1$. In particular, for any $\theta \in (0, \pi/2)$ there is a unique $\delta \in (0,1)$ such that $\mathcal{Y}_\theta^{\blacktriangleleft} = \mathcal{Y}_{\nu_1 \mathrm{e}_1, \delta}^{\blacktriangleleft}$ where

$$\mathcal{Y}_\theta^{\blacktriangleleft} = \{x \in \mathbb{R}^2 : \ \sin(\theta) x_1 \geq |x_2|\}$$

is the symmetric cone with angular aperture 2θ. Similarly to section 7, say that x is a θ-cone-point of A if $A \subset x + (\mathcal{Y}_\theta^{\blacktriangleleft} \cup \mathcal{Y}_\theta^{\blacktriangleright})$. We (re-)introduce

$$\mathcal{Y}_\theta^{\blacktriangleright} = -\mathcal{Y}_\theta^{\blacktriangleleft}, \quad \mathrm{Diamond}_\theta(u,v) = (u + \mathcal{Y}_\theta^{\blacktriangleleft}) \cap (v + \mathcal{Y}_\theta^{\blacktriangleright}),$$
$$\mathrm{CPts}_\theta(A) = \{x \in A : \ x \text{ is a } \theta\text{-cone-point of } A\}.$$

When omitted from the notation, θ is set to be $\pi/4$. Denote $\mathfrak{B}_L, \mathfrak{B}_R, \mathfrak{A}$ the sets of backward-confined, forward-confined, diamond-confined graphs (see section 7.1). Let then p_L, p_R, p be the measures given by Theorem 9 for $(\delta, \nu_1 \mathrm{e}_1)$ with δ associated to $\theta = \pi/4$. Denote $\mathcal{Q}$ the positive measure (*not* probability measure) on length + sequences given by

$$\mathcal{Q}(M; \gamma_L, \gamma_1, \dots, \gamma_M, \gamma_R) = \mathrm{C} p_L(\gamma_L) p_R(\gamma_R) \prod_{i=1}^{M} p(\gamma_i)$$

where C is the constant given by Theorem 9. If is worth stressing that whenever f vanishes when the displacement of $\gamma_L \circ \gamma_1 \circ \dots \circ \gamma_M \circ \gamma_R$ becomes larger than some constant, integrating f against $\mathcal{Q}$ is just a finite sum. We will write

$$\int d\mathcal{Q} f(M; \gamma_L, \gamma_1, \dots, \gamma_M, \gamma_R)$$
$$= \mathrm{C} \sum_{M \geq 0} \sum_{\gamma_L \in \mathfrak{B}_L} \sum_{\gamma_R \in \mathfrak{B}_R} \sum_{\gamma_1, \dots, \gamma_M \in \mathfrak{A}} f(M; \gamma_L, \dots, \gamma_R) p_L(\gamma_L) p_R(\gamma_R) \prod_{i=1}^{M} p(\gamma_i). \quad (46)$$

Let $\gamma_1, \gamma_2, \dots$ be an i.i.d. sequence of random diamond-confined graphs with law p, and define

$$T_v = \inf\{k \geq 1 : \ \mathrm{X}(\gamma_1 \circ \dots \circ \gamma_k) = v\},$$

where the inf of an empty set is set to be $+\infty$ by convention. Let Q_v be the law of $(\gamma_1, \gamma_2, \dots, \gamma_{T_v})$ conditioned on $\{T_v < \infty\}$.

The main Theorem of this section is the following "representation as a mixture of random walk bridges".

Theorem 11. *There are $c > 0, C_0 \geq 0, n_0 \geq 1$ such that for any $n \geq n_0$, $m \geq C_0 n$, there is a probability measure $\overline{\mathrm{Me}}_{n,m}$ on $\mathfrak{B}_L \times \mathfrak{B}_R$ such that*

(1) $\overline{\mathrm{Me}}_{n,m}$ is supported on pairs of graphs with displacement sup-norm at most $2\ln^9(n)$;

(2) for any f function of $\mathcal{C}$,

$$\Big| \sum_{\zeta_L, \zeta_R} \overline{\mathrm{Me}}_{n,m}(\zeta_L, \zeta_R) E_{Q_{v(\zeta_L,\zeta_R)}}\big(f(v_L + \zeta_L \circ \gamma_1 \circ \dots \circ \gamma_M \circ \zeta_R)\big)$$
$$- \Phi_m\big(f(\mathcal{C}) \,\big|\, v_L \leftrightarrow v_R\big) \Big| \leq \|f\|_\infty e^{-c \ln^3(n)}$$

where $(\gamma_1, \gamma_2, \dots, \gamma_M) \sim Q_{v(\zeta_L,\zeta_R)}$, E_{Q_v} denotes expectation with respect to Q_v, and $v(\zeta_L, \zeta_R) = v_R - \mathrm{X}(\zeta_R) - v_L - \mathrm{X}(\zeta_L)$.

8.2. **Basic properties of the measure.** We start with some general observations on Φ_m before going to the heart of the argument. As we will use this quantity several times, it is worthwhile to define

$$\tilde{n} = n - \lfloor \ln^2(n) \rfloor.$$

Lemma 8.1. *There are $c > 0, C \geq 0$ such that for every $n, m \geq 2$, any $F \subset \mathbb{E}_{\Lambda_{n-1,m-1}}$, and any event A supported on F with $\Phi(A) > 0$,*

$$\Big|\frac{\Phi_m(A)}{\Phi(A)} - 1\Big| \leq C \sum_{e \in F} e^{-c\mathrm{d}_\infty(e, \Lambda_{n-1,m-1}^c)}.$$

Proof. This follows from the stochastic domination (45) and from the exponential ratio weak mixing of ATRC (Theorem 4). □

Recall that a subset F of $\mathbb{E}$ is *lattice simply connected* if both $(\mathbb{V}_F, F)$ and the planar dual of $(\mathbb{V}_{F^c}, F^c)$ are connected. We import the following result of [Ale04].

Theorem 12 (Alexander, 2004)**.** *There is $c > 0$ such that for any simply connected $F \subset \mathbb{E}$, and any $x, y \in \mathbb{V}_F$,*

$$\sup_\eta \Phi\big(x \overset{F}{\longleftrightarrow} y \,\big|\, \xi_\tau(e) = \eta(e) \,\forall e \in F^c\big) \leq e^{-c|x-y|}.$$

Proof. This is [Ale04, Theorem 1.1]: the push-forward of Φ by ξ_τ is translation invariant, has finite energy for closing edges, exponential decay of connectivities, and is exponentially weak mixing. □

From this result, we can deduce the exponential decay for Φ_m.

Lemma 8.2. *There is $c > 0$ such that for any $n, m \geq 1$, any simply connected $F \subset E \cup E_b^+$, and any $x, y \in \mathbb{V}_F$,*

$$\sup_\eta \Phi_m\big(x \overset{F}{\longleftrightarrow} y \,\big|\, \xi_\tau(e) = \eta(e) \,\forall e \in F^c\big) \leq e^{-c|x-y|}.$$

In particular, there is an integer $C_0 \geq 0$ such that for any $m \geq C_0 n$,

$$\Phi_m(\exists x \in \mathcal{C} : \; |x_2| \geq C_0 n) \leq e^{-3\nu_1 n}.$$

Proof. The second point follows from the first and a union bound. We focus on the first. The claim with $\Phi = \mathsf{ATRC}$ replacing Φ_m is Theorem 5. In particular, by the stochastic domination (45), for any $F \subset E$,

$$\Phi_m\big(\,\cdot\,\big|\,\xi_\tau(e) = \eta(e) \,\forall e \in F^c\big) \preccurlyeq \mathsf{ATRC}_F^{1,1} = \Phi(\,\,|\,\xi_\tau(e) = 1 \,\forall e \in F^c),$$

so the first equation holds for Φ_m when $F \cap E_b^+ = \varnothing$. Let us prove the case where $F \cap E_b^+ = I \neq \varnothing$. □

Let $R_{k,l} = \{-k, \dots, k\} \times \{-l, \dots, l\}$. Define

- $\mathrm{Cross}_{k,l}^{\geq 2}$ the event that there is at least two disjoint clusters in $(R_{l,k}, \xi_\tau \cap \mathbb{E}_{R_{l,k}})$ crossing $R_{k,l}$ from left to right;
- $\mathrm{Cross}_{k,l}^{1}$ the event that there is exactly one cluster in $(R_{l,k}, \xi_\tau \cap \mathbb{E}_{R_{l,k}})$ crossing $R_{k,l}$ from left to right.

Lemma 8.3. *There is $c > 0, n_0 \geq 1$, such that for any $n \geq n_0$, $m \geq \ln^2(n)$, $k \leq \min(m - \ln^2(n), n^2)$, one has*

$$\Phi_m\big(\mathrm{Cross}_{\tilde{n},k}^{\geq 2}\big) \leq 2(2k+1)^2 e^{-\nu_1 2n - cn},$$

$$\Phi_m\big(\mathrm{Cross}_{\tilde{n},k}^{1}\big) \leq 2(2k+1)^2 e^{-\nu_1 2\tilde{n}}.$$

Proof. Note that the events $\mathrm{Cross}^{\geq 2}_{\tilde n,k}$, $\mathrm{Cross}^{1}_{\tilde n,k}$ are supported on $F_k \equiv \mathbb{E}_{R_{\tilde n,k}}$. Given the constraints on k, one can apply Lemma 8.1 to reduce the problem to proving the same bound for Φ instead of Φ_m (the cost for passing from Φ_m to Φ is the factor 2 in the claim). Now, by a union bound,

$$\Phi(\mathrm{Cross}^{\geq 2}_{\tilde n,k}) \leq \sum_{-k\leq x_2<u_2\leq k}\ \sum_{-k\leq y_2<v_2\leq k} \Phi(x \xleftrightarrow{F_k} y \overset{F_k}{\not\leftrightarrow} u \xleftrightarrow{F_k} v)$$

where connections are understood in ξ_τ, and $x=(-\tilde n,x_2)$, $u=(-\tilde n,u_2)$, $y=(\tilde n,y_2)$, and $v=(\tilde n,v_2)$. Denote then $\mathcal{C}_{k,x}$ the cluster of x in the restriction of ξ_τ to F_k, and, for C a realisation of $\mathcal{C}_{k,x}$ with $y\in C$, $\mathcal{F}^+_k(C)$ the set of edges in F_k above the union of C and its boundary. $\mathcal{F}^+_k(C)$ is then a simply connected set, so by Theorem 5,

$$\begin{aligned}\Phi(x \xleftrightarrow{F_k} y \overset{F_k}{\not\leftrightarrow} u \xleftrightarrow{F_k} v) &\leq \sum_{\substack{C\subset R_{\tilde n,k}\\ x,y\in C}} \Phi(\mathcal{C}_{k,x}=C)\mathsf{ATRC}^{1,1}_{\mathcal{F}^+_k(C)}(u\leftrightarrow v)\\ &\leq e^{-c|u-v|}\Phi(x\leftrightarrow y) \leq e^{-cn}e^{-\nu(y-x)}.\end{aligned}$$

Now, by convexity and symmetry, $\nu(y-x)\geq 2\tilde n\nu((y-x)/|y-x|)\geq 2\tilde n\nu_1$, so for n large enough,

$$\Phi(\mathrm{Cross}^{\geq 2}_{\tilde n,k}) \leq (2k+1)^2 e^{-cn}e^{-2n\nu_1},$$

which is the first half of the claim. The second follows from the same argument:

$$\Phi(\mathrm{Cross}^{1}_{\tilde n,k}) \leq \sum_{-k\leq x_2,y_2\leq k} \Phi(x \xleftrightarrow{F_k} y) \leq (2k+1)^2\Phi(x\leftrightarrow y) \leq (2k+1)^2 e^{-2\tilde n\nu_1}.$$

□

Lemma 8.4. *There is $c\geq 0$ such that for any $\epsilon>0$ there is $n_0\geq 1$ such that for any $n\geq n_0$, $m\geq \epsilon n+\ln^2(n)$,*

$$\Phi_m(v_L\leftrightarrow v_R)\geq e^{-c\ln^2(n)}e^{-2n\nu_1}.$$

Proof. Let $w_L=(-\tilde n,0)$, $w_R=(\tilde n,0)$. Then, by inclusion of events and FKG inequality,

$$\begin{aligned}\Phi_m(v_L\leftrightarrow v_R) &\geq \Phi_m(w_L\leftrightarrow v_L)\Phi_m(w_R\leftrightarrow v_R)\Phi_m(w_L \xleftrightarrow{R_{\tilde n,\lfloor\epsilon n\rfloor}} w_R)\\ &\geq 2\Phi(w_L \xleftrightarrow{R_{\tilde n,\lfloor\epsilon n\rfloor}} w_R)\Phi_m(w_L\leftrightarrow v_L)\Phi_m(w_R\leftrightarrow v_R)\end{aligned}$$

where we used Lemma 8.1 and n large enough in the second line.

We can then use Theorem 9 to lower bound the first probability: $\Phi(w_L \xleftrightarrow{R_{\tilde n,\lfloor\epsilon n\rfloor}} w_R)$ is equal to $e^{-2\nu_1\tilde n}$ times the probability for a random walk with step distribution supported on the whole of $\mathcal{Y}^{◂}\cap\mathbb{Z}^2$, having exponential tails, and mean proportional to e_1 to hit w_R-w_L while having second component less (in modulus) than ϵn. This has probability greater than $C/\sqrt{n}$ (and we only need $e^{-c\ln^2(n)}$).

Finally, we use finite energy to bound the last two probabilities: let $F_L=\big\{\{x,x+\mathrm{e}_1\}:\ x_1=-n-1,\dots,\tilde n-1\big\}$. Opening the edges of F_L has probability at least $e^{-c|F_L|}=e^{-c\ln^2(n)}$ for some $c>0$. So $\Phi_m(w_L\leftrightarrow v_L)\geq e^{-c\ln^2(n)}$. Same with $w_R\leftrightarrow v_R$. □

This leads us to a first notion of "good configurations":

$$\mathrm{Good}^1_n=\{v_L\leftrightarrow v_R\}\cap\mathrm{Cross}^1_{\tilde n,C_0 n}\cap\{\forall x\in\mathcal{C}:\ |x_2|<C_0 n\},\tag{47}$$

where C_0 is given by Lemma 8.2. Then, by Lemmas 8.4, 8.3, and 8.2, there is $c > 0, n_0 \geq 1$ such that for any $n \geq n_0$, $m \geq C_0 n + \ln^2(n)$,

$$\Phi_m(\mathrm{Good}_n^1 \,|\, v_L \leftrightarrow v_R) \geq 1 - e^{-cn}. \tag{48}$$

Define $\mathcal{C}_{\mathrm{cr}}$ to be the unique crossing cluster of $R_{\tilde{n},C_0 n}$ when there is a unique cluster, and $\mathcal{C}_{\mathrm{cr}} = R_{\tilde{n},C_0 n}$ else. Under Good_n^1, one has $\mathcal{C}_{\mathrm{cr}} \subset \mathcal{C}$. The general plan from now on will be to study $\mathcal{C}$ by first proving that it is close (in Hausdorff distance) to $\mathcal{C}_{\mathrm{cr}}$, and then by studying $\mathcal{C}_{\mathrm{cr}}$ using the infinite volume study of section 7.

8.3. **Geometry of crossing cluster.** We start with the proximity between $\mathcal{C}_{\mathrm{cr}}$ and $\mathcal{C}$. Introduce

$$\mathcal{L}_L(C) = C \cap \{-\tilde{n}\} \times \mathbb{Z}, \quad \mathcal{L}_R(C) = C \cap \{\tilde{n}\} \times \mathbb{Z}, \quad \mathcal{L}(C) = \mathcal{L}_L(C) \cup \mathcal{L}_R(C).$$

Lemma 8.5. *Let C_0 be given by Lemma 8.2. There is $c > 0, n_0 \geq 1$, such that for any $n \geq n_0$, $m \geq C_0 n + \ln^2(n)$, one has*

$$\Phi_m\big(\exists x \in \mathcal{C} \setminus \mathcal{C}_{\mathrm{cr}} :\ \mathrm{d}_\infty(x, \mathcal{L}(\mathcal{C}_{\mathrm{cr}})) \geq \ln^3(n) \,\big|\, \mathrm{Good}_n^1\big) \leq e^{-c\ln^3(n)}.$$

Proof. Let $C \neq \varnothing$ be a realisation of $\mathcal{C}_{\mathrm{cr}}$. Now, under the event $\mathcal{C}_{\mathrm{cr}} = C$, $x \notin C$ is connected to C only if x is connected to $\mathcal{L}_L(C) \cup \mathcal{L}_R(C)$. So, by Lemma 8.2,

$$\Phi_m(x \leftrightarrow C \,|\, \mathcal{C}_{\mathrm{cr}} = C) \leq \sum_{y \in \mathcal{L}(C)} e^{-c|\mathrm{d}_\infty(x,y)|}.$$

In particular, for C a realisation of $\mathcal{C}_{\mathrm{cr}}$ such that $C \subset R_{\tilde{n},C_0 n}$, and n large enough,

$$\begin{aligned}\Phi_m(\exists x :\ \mathrm{d}(x, C) \geq \ln^3(n), x \leftrightarrow C \,|\, \mathcal{C}_{\mathrm{cr}} = C)& \\ \leq \sum_{y \in \mathcal{L}(C)} \sum_{x \in \mathbb{Z}^2 : \mathrm{d}_\infty(x,y) \geq \ln^3(n)} & e^{-c|\mathrm{d}_\infty(x,y)|} \leq e^{-c\ln^3(n)/2},\end{aligned}$$

as $|\mathcal{L}(C)| \leq 2C_0 n$. In particular,

$$\begin{aligned}&\Phi_m\big(\exists x :\ \mathrm{d}(x, C) \geq \ln^3(n), x \leftrightarrow C \,|\, \mathrm{Cross}^1_{\tilde{n},C_0 n} \cap \{\forall x \in \mathcal{C}_{\mathrm{cr}} :\ |x_2| < C_0 n\}\big) \\ &\quad \leq \sum_C^{*} \Phi_m\big(\mathcal{C}_{\mathrm{cr}} = C \,|\, \mathrm{Cross}^1_{\tilde{n},C_0 n} \cap \{\forall x \in \mathcal{C}_{\mathrm{cr}} :\ |x_2| < C_0 n\}\big) e^{-c\ln^3(n)/2} \\ &\quad = e^{-c\ln^3(n)/2}.\end{aligned}$$

where $\sum_C^*$ is over realisation of $\mathcal{C}_{\mathrm{cr}}$ such that $C \subset R_{\tilde{n},C_0 n}$. To conclude,

$$\begin{aligned}&\frac{\Phi_m\big(\exists x \in \mathcal{C} \setminus \mathcal{C}_{\mathrm{cr}} :\ \mathrm{d}_\infty(x, \mathcal{L}(\mathcal{C}_{\mathrm{cr}})) \geq \ln^3(n), \mathrm{Good}_n^1\big)}{\Phi_m(\mathrm{Good}_n^1)} \\ &\quad \leq \frac{e^{-c\ln^3(n)} \Phi_m(\mathrm{Cross}^1_{\tilde{n},C_0 n} \cap \{\forall x \in \mathcal{C}_{\mathrm{cr}} :\ |x_2| < C_0 n\})}{\Phi_m(\mathrm{Good}_n^1)} \leq e^{-c\ln^3(n)} e^{c'\ln^2(n)}\end{aligned}$$

where the numerator is bounded using the bound we just derived, inclusion of events, and Lemma 8.3, and the denominator is bounded using Lemma 8.4, and (48). □

We then turn to the geometry of $\mathcal{C}_{\mathrm{cr}}$. Introduce

$$\mathrm{Good}_n^2 = \{\forall x \in \mathcal{C} \setminus \mathcal{C}_{\mathrm{cr}} :\ \mathrm{d}_\infty(x, \mathcal{L}(\mathcal{C}_{\mathrm{cr}})) \geq \ln^3(n)\} \cap \mathrm{Good}_n^1.$$

By Lemma 8.5,

$$\Phi_m(\mathrm{Good}_n^2 \,|\, \mathrm{Good}_n^1) \geq 1 - e^{-c\ln^3(n)}. \tag{49}$$

Introduce the slabs: for $l \in \mathbb{Z}, k \geq 0$,

$$\mathcal{S}_{l,k} = \{l, l+1, \dots, l+k\} \times \mathbb{Z}.$$

Lemma 8.6. *Let C_0 be given by Lemma 8.2. There are $c > 0, n_0 \geq 1$ such that for any $n \geq n_0$, $m \geq C_0 n + \ln^2(n)$,*

$$\Phi_m\big(\exists k \geq \ln^4(n), -n-1 \leq l \leq n+1-k : \ \mathcal{S}_{l,k} \cap \mathrm{CPts}(\mathcal{C}) = \varnothing \,\big|\, \mathrm{Good}_n^2\big) \leq e^{-c\ln^3(n)}.$$

Proof. Let

$$A_L = \{\mathcal{L}_L(\mathcal{C}_{\mathrm{cr}}) \cap \Lambda_{\lceil \ln^3(n)\rceil}(v_L) \neq \varnothing\}, \quad A_R = \{\mathcal{L}_R(\mathcal{C}_{\mathrm{cr}}) \cap \Lambda_{\lceil \ln^3(n)\rceil}(v_L) \neq \varnothing\},$$
$$A = A_L \cap A_R \cap \mathrm{Cross}^1_{\tilde{n},C_0 n} \cap \{\forall x \in \mathcal{C}_{\mathrm{cr}} : \ |x_2| < C_0 n\},$$

where $\Lambda_l(x) = x + \{-l, \dots, l\}^2$. Start by proving that for any $K_n \geq \ln^3(n)$ (and n large enough)

$$\Phi_m\big(\exists k \geq K_n, \tilde{n} \leq l \leq \tilde{n} : \ \mathcal{S}_{l,k} \cap \mathrm{CPts}(\mathcal{C}_{\mathrm{cr}}) = \varnothing, A\big) \leq e^{-cK_n} e^{-2n\nu_1}. \tag{50}$$

By Lemma 8.1 and a union bound, the above probability is upper bounded by (for n large enough),

$$2 \sum_{x_2,y_2 = -\lceil \ln^3(n)\rceil}^{\lceil \ln^3(n)\rceil} \Phi\big(\exists k \geq K_n, \tilde{n} \leq l \leq \tilde{n} : \ \mathcal{S}_{l,k} \cap \mathrm{CPts}(\mathcal{C}_x) = \varnothing, x \leftrightarrow y\big)$$

where $x = (-\tilde{n}, x_2)$, $y = (\tilde{n}, y_2)$. By Theorem 9 (exponential tails for the size of the steps gives exponential tails for the distance between cone-points), for every x, y as in the above sum,

$$e^{2\tilde{n}\nu_1} \Phi\big(\exists k \geq K_n, \tilde{n} \leq l \leq \tilde{n} : \ \mathcal{S}_{l,k} \cap \mathrm{CPts}(\mathcal{C}_x) = \varnothing, x \leftrightarrow y\big) \leq e^{-cK_n}.$$

Plugging this in the previous display gives (50). Now, for n large enough, the event

$$\{\exists k \geq \ln^4(n), -n-1 \leq l \leq n+1-k : \ \mathcal{S}_{l,k} \cap \mathrm{CPts}(\mathcal{C}) = \varnothing\} \cap \mathrm{Good}_n^2$$

is included in the event $\{\exists k \geq \ln^3(n), \tilde{n} \leq l \leq \tilde{n} : \ \mathcal{S}_{l,k} \cap \mathrm{CPts}(\mathcal{C}_{\mathrm{cr}}) = \varnothing\} \cap A$ (the cones grove linearly, so, under Good_n^2, the presence of cone-points of $\mathcal{C}_{\mathrm{cr}}$ in $\mathcal{S}_{-n+3\ln^4(n)/4, \ln^4(n)/4}$ and in $\mathcal{S}_{-n+\ln^4(n)/4, \ln^4(n)/4}$ implies the presence of a cone-point of $\mathcal{C}$ in $\mathcal{S}_{-n+3\ln^4(n)/4, \ln^4(n)/4}$). In particular, one has (for n large enough)

$$\frac{\Phi_m\big(\exists k \geq \ln^4(n), -n-1 \leq l \leq n+1-k : \ \mathcal{S}_{l,k} \cap \mathrm{CPts}(\mathcal{C}) = \varnothing, \mathrm{Good}_n^2\big)}{\Phi_m\big(\mathrm{Good}_n^2\big)}$$
$$\leq e^{-2\tilde{n}\nu_1} e^{-c\ln^3(n)} 2 e^{2n\nu_1} e^{c\ln^2(n)} \leq e^{-c\ln^3(n)/2}$$

where the numerator is upper bounded using what we just derived, and the denominator is lower bounded using Lemma 8.4, (48), and (49). □

Finally, introduce

$$\mathrm{Good}_n^3 = \{\forall k \geq \ln^4(n), -n-1 \leq l \leq n+1-k : \ \mathcal{S}_{l,k} \cap \mathrm{CPts}(\mathcal{C}) \neq \varnothing\} \cap \mathrm{Good}_n^2.$$

By Lemma 8.6, (48), and (49),

$$\Phi_m(\mathrm{Good}_n^3 \,|\, v_L \leftrightarrow v_R) \geq 1 - e^{-c\ln^3(n)}. \tag{51}$$

Note that all the arguments we made to obtain the above bound work exactly in the same fashion (are even substantially easier) for Φ.

8.4. **Density swapping.** Under Good_n^3, the slabs

$$\mathcal{S}_{-n+2\lceil \ln^4(n)\rceil,\lceil \ln^4(n)\rceil},\quad \mathcal{S}_{n-3\lceil \ln^4(n)\rceil,\lceil \ln^4(n)\rceil},$$

both contain a cone-point of $\mathcal{C}$. Let

- $W_L = W_L(\mathcal{C})$ be the leftmost cone-point of $\mathcal{C}$ in the slab $\mathcal{S}_{-n+2\lceil \ln^4(n)\rceil,\lceil \ln^4(n)\rceil}$,
- $W_R = W_R(\mathcal{C})$ be the rightmost cone-point of $\mathcal{C}$ in the slab $\mathcal{S}_{n-3\lceil \ln^4(n)\rceil,\lceil \ln^4(n)\rceil}$.

One can then uniquely decompose a realisation C of $\mathcal{C}$ contributing to Good_n^3 as

$$C = v_L + \eta_L \circ \eta \circ \eta_R$$

with $\eta_L \in \mathfrak{B}_L$, $\eta_R \in \mathfrak{B}_R$, $\eta \in \mathfrak{A}$, $\mathrm{X}(\eta_L) = W_L(C) - v_L$, $\mathrm{X}(\eta_R) = v_R - W_R(C)$. Now, notice that by the cone constraint, $\mathbf{b}(\eta_L)$ has degree one in η_L, $\mathbf{b}(\eta_R)$ has degree one in η_R, and $\mathbf{f}(\eta), \mathbf{b}(\eta)$ both have degree 1 in η. Let

$$\mathbf{f}_e(\eta) = \{\mathbf{f}(\eta), \mathbf{f}(\eta) + \mathrm{e}_1\},\quad \mathbf{b}_e(\eta) = \{\mathbf{b}(\eta), \mathbf{b}(\eta) - \mathrm{e}_1\},$$
$$\mathbf{f}_e(\eta_R) = \{\mathbf{f}(\eta_R), \mathbf{f}(\eta_R) + \mathrm{e}_1\},\quad \mathbf{b}_e(\eta_L) = \{\mathbf{b}(\eta_L), \mathbf{b}(\eta_L) - \mathrm{e}_1\}.$$

Introduce percolation events associated with $\eta_L \in \mathfrak{B}_L, \eta_R \in, \eta$: let $u_R = v_R - \mathrm{X}(\eta_R)$ and define

- $A_L(\eta_L)$ is the event that edges of $v_L + \eta_L$ are open in $\xi_{\tau\tau'}$, edges of $(v_L + \eta_L) \setminus \mathbf{b}_e(v_L + \eta_L)$ are open in ξ_τ, edges of $(v_L + \partial^{\mathrm{ex}}\eta_L) \cap (E \cup E_b^+)$ are closed in ξ_τ, $\mathbf{b}_e(v + \eta_L)$ is closed in ξ_τ;
- $A_R(\eta_R)$ is the event that edges of $u_R + \eta_R$ are open in $\xi_{\tau\tau'}$, edges of $(u_R + \eta_R) \setminus \mathbf{f}_e(u_R + \eta_R)$ are open in ξ_τ, edges of $(u_R + \partial^{\mathrm{ex}}\eta_R) \cap (E \cup E_b^+)$ are closed in ξ_τ, $\mathbf{f}_e(u_R + \eta_R)$ is closed in ξ_τ;
- $A_v(\eta)$ is the event that edges of $v + \eta$ are open in $\xi_{\tau\tau'}$, edges of $(v + \eta) \setminus \{\mathbf{b}_e(v + \eta), \mathbf{f}_e(v + \eta)\}$ are open in ξ_τ, edges of $\partial^{\mathrm{ex}}(v + \eta)$ are closed in ξ_τ, $\mathbf{b}_e(v + \eta), \mathbf{f}_e(v + \eta)$ are closed in ξ_τ.

Using the same trick as in section 7.6 (swapping the state of $\mathbf{f}_e(\eta), \mathbf{b}_e(\eta), \mathbf{f}_e(\eta_R), \mathbf{b}_e(\eta_L)$ from open to close in ξ_τ brings a weight $\frac{2^4 2^1}{2^0 2^5} = 1$: four open edges giving one connected component, versus zero open edges and five connected components: the left, middle, and right graphs as well as the two cone-points that become isolated), one gets

$$\Phi_m(\mathcal{C} = v_L + \eta_L \circ \eta \circ \eta_R) = \Phi_m\big(A_L(\eta_L) \cap A_{v_L + \mathrm{X}(\eta_L)}(\eta) \cap A_R(\eta_R)\big).$$

Now, define the infinite volume events corresponding to the pieces η_L, η_R:

- $\tilde{A}_L(\eta_L)$ is the event that edges of $v_L + \eta_L$ are open in $\xi_{\tau\tau'}$, edges of $(v_L + \eta_L) \setminus \mathbf{b}_e(v_L + \eta_L)$ are open in ξ_τ, edges of $v_L + \partial^{\mathrm{ex}}\eta_L$ are closed in ξ_τ, $\mathbf{b}_e(v + \eta_L)$ is closed in ξ_τ;
- $\tilde{A}_R(\eta_R)$ is the event that edges of $u_R + \eta_R$ are open in $\xi_{\tau\tau'}$, edges of $(u_R + \eta_R) \setminus \mathbf{f}_e(u_R + \eta_R)$ are open in ξ_τ, edges of $u_R + \partial^{\mathrm{ex}}\eta_R$ are closed in ξ_τ, $\mathbf{f}_e(u_R + \eta_R)$ is closed in ξ_τ.

Let C_0 be given by Lemma 8.2. Define, for $i = 1, 2, 3$

$$\Delta_n^i = \{-n + i\lceil \ln^4(n)\rceil, \dots n - i\lceil \ln^4(n)\rceil\} \times \{-C_0 n + i\lceil \ln^4(n)\rceil, \dots C_0 n - i\lceil \ln^4(n)\rceil\}.$$

In particular, for any $C = v_L + \eta_L \circ \eta \circ \eta_R$ contributing to Good_n^3, $W_L(C), W_R(C) \in \Delta_n^2 \setminus \Delta_n^3$, and $W_L(C) + \eta$ is a subset of the edges in $\mathbb{E}_{\Delta_n^2}$.

Lemma 8.7 (Density swapping). *There are $c > 0, n_0 \geq 1$ such that for any $n \geq n_0$, $m \geq C_0 n + \ln^2(n)$, any $\eta_L \in \mathfrak{B}_L$, $\eta_R \in \mathfrak{B}_R$ with $\{v_L + \mathrm{X}(\eta_L), v_R - \mathrm{X}(\eta_R)\} \subset \Delta_n^2 \setminus \Delta_n^3$,* *and any event B with support in $\mathbb{E}_{\Delta_n^2}$, one has*

$$\Big|\frac{\Phi_m\big(B\,|\,A_L(\eta_L)\cap A_R(\eta_R)\big)}{\Phi\big(B\,|\,\tilde{A}_L(\eta_L)\cap\tilde{A}_R(\eta_R)\big)}-1\Big|\leq e^{-c\ln^4(n)}.$$

Proof. This follows by standard mixing arguments, Lemma A.1 (ratio mixing) and finding contours in $\Delta_n^1\setminus\Delta_n^2$. □

Introduce the set of good clusters

$$\mathrm{GCl}=\{C\ni v_l,v_R:\ C\subset V_{n,m},\ C=v_L+\eta_L\circ\eta\circ\eta_R\}$$

with $\eta\in\mathfrak{A}$, $\eta_L\in\mathfrak{B}_L$, $\eta_R\in\mathfrak{B}_R$ so that $W_L(C)=v_L+\mathrm{X}(\eta_L)$, $W_R(C)=v_R-\mathrm{X}(\eta_R)$.

One directly has

$$\mathrm{Good}_n^3\subset\{\mathcal{C}\in\mathrm{GCl}\}. \tag{52}$$

For η_L,η_R as before, introduce the shorthand $r_m(\eta_L,\eta_R)\equiv\Phi_m\big(A_L(\eta_L)\cap A_R(\eta_R)\big)$. Looking back at what we obtained: for any f function of $\mathcal{C}$ with $\|f\|_\infty\leq 1$, we obtain by applying Lemma 8.7

$$\Big|\Phi_m\big(f(\mathcal{C})\mathbb{1}_{\mathrm{GCl}}(\mathcal{C})\big)-\sum_{\eta_L,\eta_R}r_m(\eta_L,\eta_R)\Phi\big(f(\mathcal{C})\mathbb{1}_{\mathrm{GCl}}(\mathcal{C})\mathbb{1}_{\mathcal{C}\sim\eta_L,\eta_R}\,|\,\tilde{A}_L(\eta_L)\cap\tilde{A}_R(\eta_R)\big)\Big|$$
$$\leq e^{-c\ln^4(n)}\Phi_m\big(\mathcal{C}\in\mathrm{GCl}\big), \tag{53}$$

where the sum is over η_L,η_R with the same displacement constraints as before (it is in particular less than $4\ln^4(n)$ in sup-norm), and $\mathbb{1}_{\mathcal{C}\sim\eta_L,\eta_R}$ is 1 if $\mathcal{C}=v_L+\eta_L\circ\eta\circ\eta_R$ for some diamond-confined η, and 0 else.

We now turn to the study of

$$\Phi\big(f(\mathcal{C})\mathbb{1}_{\mathrm{GCl}}(\mathcal{C})\mathbb{1}_{\mathcal{C}\sim\eta_L,\eta_R}\,|\,\tilde{A}_L(\eta_L)\cap\tilde{A}_R(\eta_R)\big)=\frac{\Phi\big(f(\mathcal{C})\mathbb{1}_{\mathcal{C}\sim\eta_L,\eta_R}\big)}{\Phi\big(\tilde{A}_L(\eta_L)\cap\tilde{A}_R(\eta_R)\big)}.$$

First, by Theorem 9 and translation invariance of Φ, there is $c>0$ independent of n such that (recall (46))

$$\Big|\int d\mathcal{Q}f(v_L+\bar{\gamma})\mathbb{1}_{v_L+\bar{\gamma}\sim\eta_L,\eta_R}-e^{2\nu_1 n}\Phi\big(f(\mathcal{C})\mathbb{1}_{\mathcal{C}\sim\eta_L,\eta_R}\big)\Big|\leq e^{-cn}, \tag{54}$$

where $\bar{\gamma}=\gamma_L\circ\gamma_1\circ\ldots\circ\gamma_M\circ\gamma_R$ (we used that if $\mathbb{1}_{v_L+\bar{\gamma}\sim\eta_L,\eta_R}=1$, then $\mathrm{X}(\bar{\gamma})=v_R-v_L$).

Introduce

$$T_L=\max\big\{k\geq 0:\ \|\mathrm{X}(\gamma_L\circ\gamma_1\circ\ldots\circ\gamma_k)\|_\infty\leq\ln^9(n)\big\},$$

and

$$T_R=\max\big\{k\geq 0:\ \|\mathrm{X}(\gamma_{M-k+1}\circ\ldots\circ\gamma_R)\|_\infty\leq\ln^9(n)\big\},$$

where $\gamma_0\equiv\gamma_L$, $\gamma_{M+1}\equiv\gamma_R$, and we put $-\infty$ for the max of an empty set by convention.

From the exponential tails of p_L,p_R,p, one obtains ($\bar{\gamma}=\gamma_L\circ\ldots\circ\gamma_R$)

$$\int d\mathcal{Q}\mathbb{1}_{T_L=-\infty}\mathbb{1}_{\mathrm{X}(\bar{\gamma})=v_R-v_L}\leq e^{-c\ln^9(n)}, \tag{55}$$

and the same for T_R, and

$$\int d\mathcal{Q}\mathbb{1}_{T_L\geq 0}\mathbb{1}_{\|\mathrm{X}(\gamma_{T_L+1})\|_\infty\geq\ln^9(n)}\mathbb{1}_{\mathrm{X}(\bar{\gamma})=v_R-v_L}\leq e^{-c\ln^9(n)}, \tag{56}$$

and the same for T_R,γ_{T_R-1}.

Let then B_n be the set of sequences $(\gamma_L,\gamma_1,\ldots,\gamma_M,\gamma_R)$ such that

- $T_R\geq 0$, $T_L\geq 0$,

- $\|\mathrm{X}(\gamma_{T_L+1})\|_\infty < \ln^9(n)$, and $\|\mathrm{X}(\gamma_{T_R-1})\|_\infty < \ln^9(n)$.

Introduce two weights that will be our new boundary weights: for $\zeta_L \in \mathfrak{B}_L$, and $\zeta_R \in \mathfrak{B}_R$, define

$$b_L(\zeta_L;\eta_L) = \sqrt{\mathrm{C}}\sum_{\gamma_L}\sum_{k\geq 1}\sum_{\gamma_1,\dots,\gamma_k} \mathbb{1}_{\|\mathrm{X}(\gamma_L\circ\dots\circ\gamma_{k-1})\|_\infty\leq \ln^9(n)}\mathbb{1}_{\|\mathrm{X}(\gamma_k)\|_\infty<\ln^9(n)}\mathbb{1}_{\exists l:\gamma_L\circ\dots\circ\gamma_l=\eta_L} \cdot \mathbb{1}_{\gamma_L\circ\dots\circ\gamma_k=\zeta_L}\cdot p_L(\gamma_L)\prod_{i=1}^k p(\gamma_i), \quad (57)$$

and

$$b_R(\zeta_R;\eta_R) = \sqrt{\mathrm{C}}\sum_{\gamma_R}\sum_{k\geq 1}\sum_{\gamma_k,\dots,\gamma_1} \mathbb{1}_{\|\mathrm{X}(\gamma_{k-1}\circ\dots\circ\gamma_1\circ\gamma_R)\|_\infty\leq \ln^9(n)}\mathbb{1}_{\|\mathrm{X}(\gamma_k)\|_\infty<\ln^9(n)} \cdot \mathbb{1}_{\exists l:\gamma_l\circ\dots\circ\gamma_1\circ\gamma_R=\eta_R}\mathbb{1}_{\gamma_k\circ\dots\circ\gamma_1\circ\gamma_R=\zeta_R}\cdot p_L(\gamma_L)\prod_{i=1}^k p(\gamma_i). \quad (58)$$

Also introduce one measure (which, after normalization, will be our Random Walk bridge measure): for $v\in\mathbb{Z}^2\setminus 0$, $M\geq 1$,

$$\tilde{Q}_v(M;\gamma_1,\dots,\gamma_M) = \mathbb{1}_{\mathrm{X}(\gamma_1\circ\dots\circ\gamma_M)=v}\prod_{i=1}^M p(\gamma_i), \quad Z_v = \int \tilde{Q}_v, \quad Q_v = \tilde{Q}_v/Z_v. \quad (59)$$

One has that b_L, b_R are supported on graphs with sup-norm less than $2\ln^9(n)$, and

$$\int d\mathcal{Q} f(v_L+\bar{\gamma})\mathbb{1}_{v_L+\bar{\gamma}\sim\eta_L,\eta_R}\mathbb{1}_{B_n}(\gamma_L,\dots,\gamma_R) = \sum_{\zeta_L\in\mathfrak{B}_L}\sum_{\zeta_R\in\mathfrak{B}_R} b_L(\zeta_L;\eta_L)b_R(\zeta_R;\eta_R)\int d\tilde{Q}_{v(\zeta_L,\zeta_R)} f(v_L+\zeta_L\circ\gamma_1\circ\dots\circ\gamma_M\circ\zeta_R). \quad (60)$$

where $v(\zeta_L,\zeta_R) = v_R - \mathrm{X}(\zeta_R) - v_L - \mathrm{X}(\zeta_L)$.

Claim 3. *With the notations above, for any f with $\|f\|_\infty\leq 1$,*

$$\Big|\sum_{\eta_L,\eta_R}\sum_{\zeta_L,\zeta_R}\frac{r_m(\eta_L,\eta_R)e^{-2\nu_1 n}b_L(\zeta_L;\eta_L)b_R(\zeta_R;\eta_R)}{\Phi\big(\tilde{A}_L(\eta_L)\cap\tilde{A}_R(\eta_R)\big)}\int d\tilde{Q}_{v(\zeta_L,\zeta_R)} f(v_L+\zeta_L\circ\gamma_1\circ\dots\circ\gamma_M\circ\zeta_R) - \Phi_m\big(f(\mathcal{C})\mathbb{1}_{\mathrm{GCl}}(\mathcal{C})\big)\Big| \leq \Phi_m\big(\mathbb{1}_{\mathrm{GCl}}(\mathcal{C})\big)e^{-c\ln^4(n)} \quad (61)$$

where the sum is over η_L,η_R with $\big\|\mathrm{X}(\eta_{L/R})\big\|_\infty\leq 4\ln^4(n)$ and ζ_L,ζ_R with $\big\|\mathrm{X}(\zeta_{L/R})\big\|_\infty\leq 2\ln^9(n)$.

Proof. Start by observing that by (51), the inclusion $\{\mathcal{C}\in\mathrm{GCl}\}\supset\mathrm{Good}_n^3$ and Lemma 8.4, one has

$$\Phi_m(\mathcal{C}\in\mathrm{GCl}) \geq e^{-2\nu_1 n}e^{-c\ln^2(n)}.$$

From (53), one has

$$\Big|\Phi_m\big(f(\mathcal{C})\mathbb{1}_{\mathrm{GCl}}(\mathcal{C})\big) - \sum_{\eta_L,\eta_R} r_m(\eta_L,\eta_R)\frac{\Phi\big(f(\mathcal{C})\mathbb{1}_{\mathcal{C}\sim\eta_L,\eta_R}\big)}{\Phi\big(\tilde{A}_L(\eta_L)\cap\tilde{A}_R(\eta_R)\big)}\Big| \leq e^{-c\ln^4(n)}\Phi_m\big(\mathcal{C}\in\mathrm{GCl}\big),$$

where the summation over the η_L, η_R are over graphs with displacement at most $4\ln^4(n)$. Then, using (54), and

$$\sum_{\eta_L,\eta_R} \frac{r_m(\eta_L,\eta_R)}{\Phi\big(\tilde{A}_L(\eta_L)\cap\tilde{A}_R(\eta_R)\big)} \leq e^{c\ln^8(n)},$$

as $\Phi\big(\tilde{A}_L(\eta_L)\cap\tilde{A}_R(\eta_R)\big) \geq e^{-c\ln^8(n)}$ by finite energy (and control over the maximal size of η_L, η_R: they contain at most $c\ln^8(n)$ edges each), one has

$$\Big| \sum_{\eta_L,\eta_R} r_m(\eta_L,\eta_R) \frac{\Phi\big(f(\mathcal{C})\mathbb{1}_{\mathcal{C}\sim\eta_L,\eta_R}\big)}{\Phi\big(\tilde{A}_L(\eta_L)\cap\tilde{A}_R(\eta_R)\big)} - \sum_{\eta_L,\eta_R} \frac{r_m(\eta_L,\eta_R)e^{-2\nu_1 n}}{\Phi\big(\tilde{A}_L(\eta_L)\cap\tilde{A}_R(\eta_R)\big)} \int d\mathcal{Q} f(v_L+\bar{\gamma})\mathbb{1}_{v_L+\bar{\gamma}\sim\eta_L,\eta_R}\Big| \leq e^{-2\nu_1 n - cn} \sum_{\eta_L,\eta_R} \frac{r_m(\eta_L,\eta_R)}{\Phi\big(\tilde{A}_L(\eta_L)\cap\tilde{A}_R(\eta_R)\big)} \leq e^{-cn/2}\Phi_m(\mathcal{C}\in\mathrm{GCl}),$$

for n large enough. Then, using (55), and (56), one obtains

$$\Big| \sum_{\eta_L,\eta_R} \frac{r_m(\eta_L,\eta_R)e^{-2\nu_1 n}}{\Phi\big(\tilde{A}_L(\eta_L)\cap\tilde{A}_R(\eta_R)\big)} \int d\mathcal{Q} f(v_L+\bar{\gamma})\mathbb{1}_{v_L+\bar{\gamma}\sim\eta_L,\eta_R}\mathbb{1}_{B_n}(\gamma_L,\dots,\gamma_R) - \sum_{\eta_L,\eta_R} \frac{r_m(\eta_L,\eta_R)e^{-2\nu_1 n}}{\Phi\big(\tilde{A}_L(\eta_L)\cap\tilde{A}_R(\eta_R)\big)} \int d\mathcal{Q} f(v_L+\bar{\gamma})\mathbb{1}_{v_L+\bar{\gamma}\sim\eta_L,\eta_R}\Big| \leq e^{-2\nu_1 n} \sum_{\eta_L,\eta_R} \frac{r_m(\eta_L,\eta_R)}{\Phi\big(\tilde{A}_L(\eta_L)\cap\tilde{A}_R(\eta_R)\big)} e^{-c\ln^9(n)} \leq e^{-c\ln^9(n)/2}\Phi_m(\mathcal{C}\in\mathrm{GCl}),$$

for n large enough. A look at (60) and triangular inequality finishes the proof. □

We are now in position to conclude the proof of Theorem 11. Define a measure Me_n on $\mathfrak{B}_L \times \mathfrak{B}_R$ via

$$\mathrm{Me}_{n,m}(\zeta_L,\zeta_R) = \mathbb{1}_{\|\mathrm{X}(\zeta_L)\|_\infty \leq 2\ln^9(n)}\mathbb{1}_{\|\mathrm{X}(\zeta_R)\|_\infty \leq 2\ln^9(n)} \cdot \frac{e^{-2\nu_1 n}}{\Phi_m(\mathcal{C}\in\mathrm{GCl})} \sum_{\eta_L,\eta_R} \frac{r_m(\eta_L,\eta_R)b_L(\zeta_L;\eta_L)b_R(\zeta_R;\eta_R)}{\Phi\big(\tilde{A}_L(\eta_L)\cap\tilde{A}_R(\eta_R)\big)} Z_{v(\zeta_L,\zeta_R)} \tag{62}$$

where the sum is over η_L, η_R with displacement less than $4\ln^4(n)$, and $v(\zeta_L,\zeta_R) = v_R - \mathrm{X}(\zeta_R) - v_L - \mathrm{X}(\zeta_L)$ (Z_v is defined in (59)). From (61) with $f = 1$, one obtains

$$\Big| \sum_{\zeta_L,\zeta_R} \mathrm{Me}_{n,m}(\zeta_L,\zeta_R) - 1 \Big| \leq e^{-c\ln^4(n)}. \tag{63}$$

Letting

$$\overline{\mathrm{Me}}_{n,m}(\zeta_L,\zeta_R) = \frac{1}{\sum_{\zeta_L,\zeta_R}\mathrm{Me}_{n,m}(\zeta_L,\zeta_R)}\mathrm{Me}_{n,m}(\zeta_L,\zeta_R), \tag{64}$$

one has the final "mixture of random walk bridges" representation: for f a function of the cluster of 0,

$$\Big|\sum_{\zeta_L,\zeta_R}\overline{\mathrm{Me}}_{n,m}(\zeta_L,\zeta_R)\int dQ_{v(\zeta_L,\zeta_R)}f(v_L+\zeta_L\circ\gamma_1\circ\ldots\circ\gamma_M\circ\zeta_R) - \Phi_m\big(f(\mathcal{C})\,\big|\,\mathcal{C}\in\mathrm{GCl}\big)\Big| \leq \|f\|_\infty e^{-c\ln^4(n)}.$$

In particular, as $\Phi_m(\mathcal{C}\in\mathrm{GCl}\,|\,v_L\leftrightarrow v_R)\geq 1-e^{-c\ln^3(n)}$, for any f,

$$\Big|\sum_{\zeta_L,\zeta_R}\overline{\mathrm{Me}}_{n,m}(\zeta_L,\zeta_R)\int dQ_{v(\zeta_L,\zeta_R)}f(v_L+\zeta_L\circ\gamma_1\circ\ldots\circ\gamma_M\circ\zeta_R) - \Phi_m\big(f(\mathcal{C})\,\big|\,v_L\leftrightarrow v_R\big)\Big| \leq \|f\|_\infty e^{-c\ln^3(n)}, \quad (65)$$

which is the statement of Theorem 11.

9. Proofs of Theorems

9.1. **Proof of Theorem 2.** We will derive Theorem 2 from the analogous statement for the cluster $\mathcal{C}_{v_L,v_R}$, Theorem 11, and the proximity statement, Lemma 4.9.

Proof of Theorem 2. To lighten the notation, we omit $p_c(q),q$ from the subscripts. Recall the definition of the upper and lower envelopes $\Gamma^{\pm,n}_{\mathsf{FK}}$ in G_n defined in Section 1, and of $\Gamma^{\pm,n,m}_{\mathsf{FK}}$ in $G=G_{n,m}$ defined analogously. We will first show the statement for the FK measure $\mathsf{FK}^{1/0}_{G_{n,Cn}}$ for C sufficiently large, and then from it derive the statement for $\mathsf{FK}^{1/0}_{G_n}$.

By Theorem 11 combined with [GI05] and Lemma 4.9, there exists $C>1$ such that the statement holds for $\Gamma^{\pm,n,Cn}_{\mathsf{FK}}$ under the measure $\mathsf{FK}^{1/0}_{G_{n,Cn}}$. The derivation for $\Gamma^{\pm,n}_{\mathsf{FK}}$ under $\mathsf{FK}^{1/0}_{G_n}$ follows from the FKG and spatial Markov properties of FK percolation, and we only sketch the argument. Define the graphs $G^{\pm}_{n,Cn}=(V^{\pm}_{n,Cn},E^{\pm}_{n,Cn})$ by

$$\Lambda^-_{n,Cn}:=\{-n,\ldots,n\}\times\{-Cn,\ldots,n\},\quad \Lambda^+_{n,Cn}:=\{-n,\ldots,n\}\times\{-n,\ldots,Cn\},$$
$$V^{\pm}_{n,Cn}=\Lambda^{\pm}_{n,Cn}\cup(\partial^{\mathrm{ex}}\Lambda^{\pm}_{n,Cn}\cap\mathbb{H}^+),\quad E^{\pm}_{n,Cn}=\mathbb{E}_{V^{\pm}_{n,Cn}}\setminus\mathbb{E}_{(\Lambda^{\pm}_{n,Cn})^c}.$$

Then, by the FKG and spatial Markov properties of the FK measures, it holds that

$$\mathsf{FK}^{1/0}_{G_{n,Cn}}\geq_{\mathrm{st}}\mathsf{FK}^{1/0}_{G^+_{n,Cn}}\leq_{\mathrm{st}}\mathsf{FK}^{1/0}_{G_n}\leq_{\mathrm{st}}\mathsf{FK}^{1/0}_{G^-_{n,Cn}}\geq_{\mathrm{st}}\mathsf{FK}^{1/0}_{G_{n,Cn}}.$$

If ω is distributed according to $\mathsf{FK}^{1/0}_{G_{n,Cn}}$, by the statement for $\Gamma^{\pm,n,Cn}_{\mathsf{FK}}$ under this measure, there exists a left-right crossing above $[-n,n]\times\{0\}$ in ω and one in ω^* below it at arbitrary small linear distance from $[-n,n]\times\{0\}$. A chain of classical monotone coupling arguments finishes the proof. $\square$

9.2. **Proof of Theorem 1.** We will derive Theorem 1 from Theorem 2. Recall the definition of the one-sided Hausdorff distance $\mathrm{d_H}$, and the notion of a subset of $\mathbb{R}^2$ being above or below a connected set in $\mathbb{L}_\bullet$ or $\mathbb{L}_\circ$, introduced above Lemma 4.9.

Proof of Theorem 1. To simplify the notation, we omit $n,q,T_c(q),p_c(q)$ from sub and superscripts. Consider the Edwards–Sokal coupling $\mathsf{ES}^{1/0}_\Lambda$ of $\sigma\sim\mathsf{Potts}^{1/\mathrm{f}}_\Lambda$ and $\omega\sim$

$\mathsf{FK}^{1/0}_{G_n}$ described in the introduction; see [Gri06, Section 1.4] for details. By Theorem 2, it suffices to show the existence of $c, C > 0$ for which, for any $k \geq 1$,

$$\mathsf{ES}^{1/0}_{\Lambda}(\mathrm{d_H}(\Gamma^{\pm}_{\mathsf{Potts}}, \Gamma^{\pm}_{\mathsf{FK}}) > k) < Cn^2 e^{-ck}. \tag{66}$$

Observe that the FK envelopes $\Gamma^{\pm}_{\mathsf{FK}}$ are deterministically above the corresponding Potts envelopes $\Gamma^{\pm}_{\mathsf{Potts}}$. In particular, $\Gamma^{+}_{\mathsf{Potts}}$ is sandwiched between $\Gamma^{-}_{\mathsf{Potts}}$ and Γ^{+}_{FK}, and it suffices to verify (66) for $\Gamma^{-}_{\mathsf{Potts}}, \Gamma^{-}_{\mathsf{FK}}$.

Let $\mathcal{C}$ be the cluster of the lower boundary in $\{\sigma \neq 1\}$, that is, the set of $i \in \Lambda$ for which there exists a path $(i_0, \dots, i_\ell)$ in Λ with $i_0 = i$, $i_\ell \in \partial^{\mathrm{in}}\Lambda \cap \mathbb{H}^-$ and $\sigma(i_k) \neq 1$ for $0 \leq k \leq \ell$. By definition, each point in the lower enevlope $\Gamma^{-}_{\mathsf{Potts}}$ is above $\mathcal{C}$. Therefore, it suffices to show that Γ^{-}_{FK} is not far above $\mathcal{C}$. Fix a realisation C of $\mathcal{C}$, and define $\Lambda_C = \Lambda \setminus (C \cup \partial^{\mathrm{ex}} C)$. By the coupling and the spatial Markov property of the Potts model, it holds that

$$\mathsf{ES}^{1/0}_{\Lambda}(\sigma|_{\Lambda_C} \in \cdot \,|\, \mathcal{C} = C) = \mathsf{Potts}^{1/\mathrm{f}}_{\Lambda}(\sigma|_{\Lambda_C} \in \cdot \,|\, \mathcal{C} = C) = \mathsf{Potts}^{1}_{\Lambda_C}.$$

Let $G_C = (V_C, E_C)$ be defined by $E_C = \mathbb{E}_{\Lambda_C} \cup \partial^{\mathrm{edge}}\Lambda_C$ and $V_C = \mathbb{V}_{E_C}$. Then, by the above and by the coupling,

$$\mathsf{ES}^{1/0}_{\Lambda}(\omega|_{E_C} \in \cdot \,|\, \mathcal{C} = C) = \mathsf{FK}^{1}_{G_C}(\omega|_{E_C} \in \cdot).$$

Now, conditional on $\mathcal{C} = C$, if $\mathrm{d_H}(C, \Gamma^{-}_{\mathsf{FK}}) > k$, then there exists a dual path of length k in $\omega^*|_{*E_C}$. Since $\mathsf{FK}^{1}_{G_C}$ stochastically dominates the infinite-volume measure FK^{1}, which admits exponential decay of connection probabilites in its dual [DGH+21, Theorem 1.2], the proof is complete. □

9.3. OZ asymptotics for the AT model.

Appendix A. Mixing to ratio mixing

We prove here a technical Lemma whose use is recurrent in the proof that mixing implies ratio mixing under suitable conditions. It is a simplified version of the argument in [Ale98, section 5].

Lemma A.1. *Let Ω_i, $i = 1, 2$ be finite sets. Let $\Omega = \Omega_1 \times \Omega_2$ and $\mathcal{F}_i = \{A \subset \Omega_i\}$, $\mathcal{F} = \{A \subset \Omega\}$. Let μ, ν be positive probability measures on $(\Omega, \mathcal{F})$. Let*

$$\pi_i : \Omega \to \Omega_i, \quad \pi_i((\omega_1, \omega_2)) = \omega_i, \quad \mu_i = \mu \circ \pi_i^{-1}.$$

Let $\epsilon_1, \epsilon_2, \epsilon_3 \in [0, 1)$. Suppose that all of the following hold:

(1) Mixing of μ, ν: for every $\xi, \xi' \in \Omega_1$, $A \subset \Omega_2$, $\rho \in \{\mu, \nu\}$

$$|\rho(\Omega_1 \times A \,|\, \{\xi\} \times \Omega_2) - \rho(\Omega_1 \times A \,|\, \{\xi'\} \times \Omega_2)| \leq \epsilon_1$$

(2) Proximity between second marginals: $\mathrm{d_{TV}}(\mu_2, \nu_2) \leq \epsilon_2$;

(3) Conditional equality: there exists an event $D \subset \Omega_2$ such that for $\rho \in \{\mu_2, \nu_2\}$, $\rho(D) \geq 1 - \epsilon_3$, and for any $y \in D$,

$$\frac{\mu(x, y)}{\mu_2(y)} = \frac{\nu(x, y)}{\nu_2(y)}, \ \forall x \in \Omega_1.$$

Then, if $\epsilon = \max(\epsilon_1, \sqrt{\epsilon_2}, \epsilon_3) \leq 0.1$, then, for any $x \in \Omega_1$,

$$1 - 9\epsilon \leq \frac{\mu_1(x)}{\nu_1(x)} \leq \frac{1}{1 - 9\epsilon}.$$

Proof. The proof goes by constructing a suitable coupling of μ, ν. Let Q be a maximal coupling of μ_2 and ν_2. Then, sample a random vector $(X, Y) = \big((X_1, X_2), (Y_1, Y_2)\big)$ as follows:

(1) sample (X_2, Y_2) using Q;
(2) sample X_1 using $\mu(\ |\,\Omega_1 \times \{X_2\}) \circ \pi_1^{-1}$;
(3) if $X_2 = Y_2 \in D$, set $Y_1 = X_1$. Else, sample $Y_1 \sim \nu(\ |\,\Omega_1 \times \{Y_2\}) \circ \pi_1^{-1}$ independently of X_1.

Denote Ψ the law of (X, Y).

Claim 4. *All the following points hold.*

(1) Ψ *is a coupling of* μ *and* ν.
(2) $(X_2 = Y_2 \in D) \implies (X_1 = Y_1)$.
(3) X_1 *and* Y_2 *are independent conditionally on* X_2, *and* Y_1 *and* X_2 *are independent conditionally on* Y_2.

Proof. The second point is by construction. We check the first point. For $x = (x_1, x_2) \in \Omega$,

$$\begin{aligned}\Psi(X = x) &= \sum_{y_2 \in \Omega_2} Q(x_2, y_2)\Psi(X_1 = x_1 \,|\, X_2 = x_2, Y_2 = y_2)\\ &= \sum_{y_2 \in \Omega_2} Q(x_2, y_2)\frac{\mu(x_1, x_2)}{\mu_2(x_2)} = \mu(x_1, x_2).\end{aligned}$$

Also, for $y = (y_1, y_2) \in \Omega$,

$$\begin{aligned}\Psi(Y = y) &= \sum_{x_2 \in \Omega_2} Q(x_2, y_2)\Psi(Y_1 = y_1 \,|\, X_2 = x_2, Y_2 = y_2)\\ &= \sum_{x_2 \in \Omega_2} Q(x_2, y_2)\Big(\mathbb{1}_{y_2 = x_2 \in D}\frac{\mu(y_1, x_2)}{\mu_2(x_2)} + (1 - \mathbb{1}_{y_2 = x_2 \in D})\frac{\nu(y_1, y_2)}{\nu_2(y_2)}\Big)\\ &= \sum_{x_2 \in \Omega_2} Q(x_2, y_2)\frac{\nu(y_1, y_2)}{\nu_2(y_2)} = \nu(y_1, y_2),\end{aligned}$$

as, by hypotheses on D, $\mathbb{1}_{y_2 = x_2 \in D}\frac{\mu(y_1, x_2)}{\mu_2(x_2)} = \mathbb{1}_{y_2 = x_2 \in D}\frac{\mu(y_1, y_2)}{\mu_2(y_2)} = \mathbb{1}_{y_2 = x_2 \in D}\frac{\nu(y_1, y_2)}{\nu_2(y_2)}$.

Let us finally check the third point. First, for any $x_1 \in \Omega_1, x_2, y_2 \in \Omega_2$,

$$\Psi(X_1 = x_1 \,|\, X_2 = x_2, Y_2 = y_2) = \frac{\mu(x_1, x_2)}{\mu_2(x_2)} = \Psi(X_1 = x_1 \,|\, X_2 = x_2).$$

Then, for any $y_1 \in \Omega_1, x_2, y_2 \in \Omega_2$,

$$\begin{aligned}\Psi(Y_1 = y_1 \,|\, X_2 = x_2, Y_2 = y_2) &= \mathbb{1}_{y_2 = x_2 \in D}\frac{\mu(y_1, x_2)}{\mu_2(x_2)} + (1 - \mathbb{1}_{y_2 = x_2 \in D})\frac{\nu(y_1, y_2)}{\nu_2(y_2)}\\ &= \frac{\nu(y_1, y_2)}{\nu_2(y_2)} = \Psi(Y_1 = y_1 \,|\, Y_2 = y_2),\end{aligned}$$

by hypotheses on D, as before. □

Introduce then $g, h : \Omega_2 \to \mathbb{R}_+$ defined by

$$\begin{aligned}g(\xi) &= \Psi(X_2 \neq Y_2 \,|\, X_2 = \xi),\\ h(\xi) &= \Psi(X_2 \neq Y_2 \,|\, Y_2 = \xi).\end{aligned}$$

Define the event H by

$$H = \{X_2 = Y_2\} \cap \{X_2 \in D\} \cap \{Y_2 \in D\} \cap \{g(X_2) \le \sqrt{\epsilon_2}\} \cap \{h(Y_2) \le \sqrt{\epsilon_2}\}.$$

Claim 5. *One has*

$$\max_{x_1 \in \Omega_2} \Psi(H^c \,|\, X_1 = x_1) \le 9\epsilon,$$
$$\max_{y_1 \in \Omega_2} \Psi(H^c \,|\, Y_1 = y_1) \le 9\epsilon.$$

Proof. Let $a = \sqrt{\epsilon_2}$. First, let $\{g > a\} = \{x_2 \in \Omega_2 :\ g(x_2) > a\}$, one has

$$\mu(\Omega_1 \times \{g > a\}) = \Psi(g(X_2) > a) \le a^{-1}\Psi(g(X_2)) = a^{-1}\Psi(\Psi(X_2 \ne Y_2 \,|\, X_2))$$
$$= a^{-1}\Psi(X_2 \ne Y_2) \le \frac{\epsilon_2}{a},$$

as $\Psi(X_2 \ne Y_2) = Q(X_2 \ne Y_2) = \mathrm{d_{TV}}(\mu_2, \nu_2) \le \epsilon_2$, and the same for $\nu(\Omega_1 \times \{h > a\})$. Now, by our first hypotheses,

$$\Psi(g(X_2) > a \,|\, X_1 = x_1) = \mu(\Omega_1 \times \{g > a\} \,|\, \{x_1\} \times \Omega_2)$$
$$\le \mu(\Omega_1 \times \{g > a\}) + \epsilon_1 = \frac{\epsilon_2}{a} + \epsilon_1,$$

and the same for $\Psi(h(Y_2) > a \,|\, Y_1 = y_1)$. Also,

$$\Psi(X_2 \in D^c \,|\, X_1 = x_1) = \mu(\Omega_1 \times D^c \,|\, \{x_1\} \times \Omega_2) \le \mu(\Omega_1 \times D^c) + \epsilon_1 = \epsilon_3 + \epsilon_1,$$

and the same for $\Psi(Y_2 \in D^c \,|\, Y_1 = y_1)$.

Then, as X_1 and Y_2 are independent conditionally on X_2 (by Claim 4),

$$\begin{aligned}
&\Psi(X_2 \ne Y_2, g(X_2) \le a \,|\, X_1 = x_1) \\
&\quad = \sum_{x_2 : g(x_2) \le a} \Psi(X_2 = x_2 \,|\, X_1 = x_1)\Psi(Y_2 \ne x_2 \,|\, X_2 = x_2) \\
&\quad = \sum_{x_2 : g(x_2) \le a} \Psi(X_2 = x_2 \,|\, X_1 = x_1) g(x_2) \le a.
\end{aligned}$$

In the same fashion, $\Psi(X_2 \ne Y_2, h(Y_2) \le a \,|\, Y_1 = y_1) \le a$. Finally,

$$\begin{aligned}
&\Psi(X_2 = Y_2 \in D, h(Y_2) > a \,|\, X_1 = x_1) \\
&\quad = \Psi(X_2 = Y_2 \in D \,|\, X_1 = x_1)\Psi(h(Y_2) > a \,|\, X_1 = x_1, X_2 = Y_2 \in D) \\
&\quad = \Psi(X_2 = Y_2 \in D \,|\, X_1 = x_1)\Psi(h(Y_2) > a \,|\, Y_1 = x_1, X_2 = Y_2 \in D) \\
&\quad = \frac{\Psi(X_2 = Y_2 \in D \,|\, X_1 = x_1)}{\Psi(X_2 = Y_2 \in D \,|\, Y_1 = x_1)}\Psi(X_2 = Y_2 \in D, h(Y_2) > a \,|\, Y_1 = x_1) \\
&\quad \le \frac{\epsilon_2/a + \epsilon_1}{1 - 2\epsilon_1 - \epsilon_3 - \epsilon_2/a - a},
\end{aligned}$$

where the second equality is because $X_2 = Y_2 \in D \implies X_1 = Y_1$, and the inequality is the previous bounds, and a union bound on $\Psi(\{X_2 = Y_2 \in D\}^c \,|\, Y_1 = x_1)$. Putting things together,

$$\begin{aligned}
\Psi(H^c \,|\, X_1 = x_1) &\le \Psi(X_2 \in D^c \,|\, X_1 = x_1) + \Psi(g(X_2) > a \,|\, X_1 = x_1) \\
&\quad + \Psi(X_2 \ne Y_2, g(X_2) \le a \,|\, X_1 = x_1) \\
&\quad + \Psi(X_2 = Y_2 \in D, h(Y_2) > a \,|\, X_1 = x_1) \\
&\le \epsilon_3 + \epsilon_1 + \frac{\epsilon_2}{a} + \epsilon_1 + a + \frac{\epsilon_2/a + \epsilon_1}{1 - 2\epsilon_1 - \epsilon_3 - \epsilon_2/a - a}.
\end{aligned}$$

One obtains the same bound on $\Psi(H^c \,|\, Y_1 = y_1)$ in the same way (with $h \leftrightarrow g$ and $Y_2 \leftrightarrow X_2$). Plugging in $a = \sqrt{\epsilon_2}$ and using the definition of ϵ gives that the last display is upper bounded by

$$5\epsilon + \frac{2\epsilon}{1-5\epsilon} \leq 9\epsilon,$$

as $\epsilon \leq 0.1$ by hypotheses. □

Let us now see how Claims 4 and 5 imply the wanted result. As Ψ is a coupling,

$$\frac{\mu_1(\xi)}{\nu_1(\xi)} = \frac{\Psi(X_1 = \xi)}{\Psi(Y_1 = \xi)} = \frac{\Psi(X_1 = \xi)\Psi(Y_1 = \xi, H)}{\Psi(Y_1 = \xi)\Psi(X_1 = \xi, H)} = \frac{\Psi(H \,|\, Y_1 = \xi)}{\Psi(H \,|\, X_1 = \xi)},$$

where the second equality is because $\{X_1 = Y_1\}$ under H (as $\{X_2 = Y_2 \in D\} \subset H$). But now,

$$1 - 9\epsilon \leq 1 - \Psi(H^c \,|\, Y_1 = \xi) \leq \frac{\Psi(H \,|\, Y_1 = \xi)}{\Psi(H \,|\, X_1 = \xi)} \leq \frac{1}{1 - \Psi(H^c \,|\, X_1 = \xi)} \leq \frac{1}{1-9\epsilon}.$$

□

References

[ADG24] Yacine Aoun, Moritz Dober, and Alexander Glazman, *Phase diagram of the Ashkin–Teller model*, Communications in Mathematical Physics **405** (2024), no. 2, 37.

[AK77] D. B. Abraham and H. Kunz, *Ornstein-Zernike theory of classical fluids at low density*, Phys. Rev. Lett. **39** (1977), no. 16, 1011–1014. MR 452115

[Ale92] Kenneth S. Alexander, *Stability of the Wulff minimum and fluctuations in shape for large finite clusters in two-dimensional percolation*, Probab. Theory Related Fields **91** (1992), no. 3-4, 507–532. MR 1151807 (93e:60191)

[Ale98] ———, *On weak mixing in lattice models*, Probab. Theory Related Fields **110** (1998), no. 4, 441–471. MR 1626951 (99e:60211)

[Ale01] Kenneth S Alexander, *Power-law corrections to exponential decay of connectivites and correlations in lattice models*, Annals of probability (2001), 92–122.

[Ale04] Kenneth S. Alexander, *Mixing properties and exponential decay for lattice systems in finite volumes*, Ann. Probab. **32** (2004), no. 1A, 441–487. MR 2040789 (2004m:60217)

[AOV24] Y. Aoun, S. Ott, and Y. Velenik, *Ornstein–Zernike behavior for Ising models with infinite-range interactions*, Annales de l'Institut Henri Poincare (B) Probabilites et statistiques **60** (2024), no. 1, 167–207.

[AT43] J. Ashkin and E. Teller, *Statistics of two-dimensional lattices with four components*, Physical Review **64** (1943), no. 5-6, 178–184.

[BD12] V. Beffara and H. Duminil-Copin, *Smirnov's fermionic observable away from criticality*, Ann. Probab. **40** (2012), no. 6, 2667–2689. MR 3050513

[BKW76] R J Baxter, S B Kelland, and F Y Wu, *Equivalence of the Potts model or Whitney polynomial with an ice-type model*, Journal of Physics A: Mathematical and General **9** (1976), no. 3, 397–406.

[CC86] J. T. Chayes and L. Chayes, *Ornstein-Zernike behavior for self-avoiding walks at all noncritical temperatures*, Comm. Math. Phys. **105** (1986), no. 2, 221–238. MR 849206

[CCC91] Massimo Campanino, JT Chayes, and L Chayes, *Gaussian fluctuations of connectivities in the subcritical regime of percolation*, Probability theory and related fields **88** (1991), no. 3, 269–341.

[CI02] Massimo Campanino and Dmitry Ioffe, *Ornstein-Zernike theory for the Bernoulli bond percolation on $\mathbb{Z}^d$*, Ann. Probab. **30** (2002), no. 2, 652–682. MR 1905854 (2003e:60216)

[CIV03] Massimo Campanino, Dmitry Ioffe, and Yvan Velenik, *Ornstein-Zernike theory for finite range Ising models above T_c*, Probab. Theory Related Fields **125** (2003), no. 3, 305–349. MR 1964456 (2005b:82016)

[CIV08] ———, *Fluctuation theory of connectivities for subcritical random cluster models*, Ann. Probab. **36** (2008), no. 4, 1287–1321. MR 2435850 (2009m:60228)

[CM97] L. Chayes and J. Machta, *Graphical representations and cluster algorithms i. discrete spin systems*, Physica A: Statistical Mechanics and its Applications **239** (1997), no. 4, 542–601.

[CS11] Dmitry Chelkak and Stanislav Smirnov, *Discrete complex analysis on isoradial graphs*, Adv. Math. **228** (2011), no. 3, 1590–1630.

[DC17] Hugo Duminil-Copin, *Lectures on the Ising and Potts models on the hypercubic lattice*, PIMS-CRM Summer School in Probability, Springer, 2017, pp. 35–161.

[DGH+21] Hugo Duminil-Copin, Maxime Gagnebin, Matan Harel, Ioan Manolescu, and Vincent Tassion, *Discontinuity of the phase transition for the planar random-cluster and Potts models with* $q > 4$, Annales Scientifiques de l'Ecole Normale Supérieure, vol. 54, SOC Mathematique France, 2021, pp. 1363–1413.

[DST17] Hugo Duminil-Copin, Vladas Sidoravicius, and Vincent Tassion, *Continuity of the phase transition for planar random-cluster and Potts models with* $1 \leq q \leq 4$, Comm. Math. Phys. **349** (2017), no. 1, 47–107.

[ES88] Robert G. Edwards and Alan D. Sokal, *Generalization of the Fortuin-Kasteleyn-Swendsen-Wang representation and Monte Carlo algorithm*, Phys. Rev. D (3) **38** (1988), no. 6, 2009–2012. MR 965465

[Fan72a] C. Fan, *On critical properties of the Ashkin-Teller model*, Physics Letters A **39** (1972), no. 2, 136.

[Fan72b] Chungpeng Fan, *Remarks on the eight-vertex model and the ashkin-teller model of lattice statistics*, Phys. Rev. Lett. **29** (1972), 158–160.

[FK72] C. M. Fortuin and P. W. Kasteleyn, *On the random-cluster model. I. Introduction and relation to other models*, Physica **57** (1972), 536–564.

[GI05] Lev Greenberg and Dmitry Ioffe, *On an invariance principle for phase separation lines*, Annales de l'Institut Henri Poincare (B) Probability and Statistics, vol. 41, Elsevier, 2005, pp. 871–885.

[GL23] Alexander Glazman and Piet Lammers, *Delocalisation and continuity in 2D: loop O(2), six-vertex, and random-cluster models*, arXiv preprint arXiv:2306.01527 (2023).

[GM23] Alexander Glazman and Ioan Manolescu, *Structure of Gibbs measures for planar FK-percolation and Potts models*, Probability and Mathematical Physics **4** (2023), no. 2, 209–256.

[GP23] Alexander Glazman and Ron Peled, *On the transition between the disordered and antiferroelectric phases of the 6-vertex model*, Electronic Journal of Probability **28** (2023), 1–53.

[Gri06] G. Grimmett, *The random-cluster model*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 333, Springer-Verlag, Berlin, 2006.

[HDJS13] Yuan Huang, Youjin Deng, Jesper Lykke Jacobsen, and Jesús Salas, *The Hintermann-Merlini-Baxter-Wu and the infinite-coupling-limit Ashkin-Teller models*, Nuclear Phys. B **868** (2013), no. 2, 492–538. MR 3006181

[Hol74] Richard Holley, *Remarks on the* FKG *inequalities*, Comm. Math. Phys. **36** (1974), 227–231. MR 0341552 (49 #6300)

[HS22] Matan Harel and Yinon Spinka, *Finitary codings for the random-cluster model and other infinite-range monotone models*, Electronic Journal of Probability **27** (2022), no. none, 1 – 32.

[Iof98] D. Ioffe, *Ornstein-Zernike behaviour and analyticity of shapes for self-avoiding walks on* $\mathbf{Z}^d$, Markov Process. Related Fields **4** (1998), no. 3, 323–350. MR 1670027 (2000d:60162)

[IOVW20] Dmitry Ioffe, Sébastien Ott, Yvan Velenik, and Vitali Wachtel, *Invariance principle for a potts interface along a wall*, Journal of Statistical Physics **180** (2020), 832–861.

[Kes82] H. Kesten, *Percolation theory for mathematicians*, Progress in Probability and Statistics, vol. 2, Birkhäuser Boston, Mass., 1982. MR 692943 (84i:60145)

[KS68] D.G. Kelly and S. Sherman, *General Griffiths's inequality on correlation in Ising ferromagnets*, J. Math. Phys. **9** (1968), 466–484.

[KW71] Leo P. Kadanoff and Franz J. Wegner, *Some critical properties of the eight-vertex model*, Phys. Rev. B **4** (1971), 3989–3993, 10.1103/PhysRevB.4.3989.

[Lis22] Marcin Lis, *Spins, percolation and height functions*, Electronic Journal of Probability **27** (2022), 1–21.

[MMSRS91] Alain Messager, Salvador Miracle-Sole, Jean Ruiz, and Senya Shlosman, *Interfaces in the potts model ii: Antonov's rule and rigidity of the order disorder interface*, Communications in mathematical physics **140** (1991), 275–290.

[MS71] L. Mittag and M. J. Stephen, *Dual transformations in many-component Ising models*, Journal of Mathematical Physics **12** (1971), no. 3, 441–450, doi:10.1063/1.1665606.

[Ott19] S. Ott, *On Ornstein-Zernike theory and some applications*, Ph.D. thesis, University of Geneva, 2019.

[Ott25] Sebastien Ott, *A new perspective on the equivalence between Weak and Strong Spacial Mixing in two dimensions*, 2025.

[OV18] Sébastien Ott and Yvan Velenik, *Potts models with a defect line*, Comm. Math. Phys. **362** (2018), no. 1, 55–106. MR 3833604

[OZ14] Leonard S Ornstein and F Zernike, *Accidental deviations of density and opalescence at the critical point of a single substance*, Proc. Akad. Sci. **17** (1914), 793.

[Pau35] Linus Pauling, *The structure and entropy of ice and of other crystals with some randomness of atomic arrangement*, Journal of the American Chemical Society **57** (1935), no. 12, 2680–2684.

[Pot52] Renfrey Burnard Potts, *Some generalized order-disorder transformations*, Proceedings of the Cambridge Philosophical Society, vol. 48(2), Cambridge Univ Press, 1952, pp. 106–109.

[PV97] C.-E. Pfister and Y. Velenik, *Random-cluster representation of the Ashkin-Teller model*, J. Statist. Phys. **88** (1997), no. 5-6, 1295–1331. MR 1478070

[Roc70] Ralph Tyrell Rockafellar, *Convex analysis*, Princeton University Press, Princeton, 1970.

[RS20] Gourab Ray and Yinon Spinka, *A short proof of the discontinuity of phase transition in the planar random-cluster model with* $q > 4$, Communications in Mathematical Physics **378** (2020), no. 3, 1977–1988.

[Rys63] Franz Rys, *Über ein zweidimensionales klassisches Konfigurationsmodell*, Helvetica Physica Acta, vol. 36, Birkhauser Verlag AG Viadukstrasse 40-44, PO Box 133, CH-4010 Basel, Switzerland, 1963, p. 537.

[Sch00] Oded Schramm, *Scaling limits of loop-erased random walks and uniform spanning trees*, Israel J. Math. **118** (2000), 221–288.

[Smi10] Stanislav Smirnov, *Conformal invariance in random cluster models. I. Holomorphic fermions in the Ising model*, Ann. of Math. (2) **172** (2010), no. 2, 1435–1467.

[Weg72] F. Wegner, *Duality relation between the Ashkin-Teller and the eight-vertex model*, Journal of physics. C, Solid state physics **5** (1972), no. 11, L131–L132.

[Wu71] F.W. Wu, *Ising model with four-spin interactions*, Phys. Rev. B **4** (1971), 2312–2314.

[Zer16] F Zernike, *The clustering-tendency of the molecules in the critical state and the extinction of light caused thereby*, Koninklijke Nederlandse Akademie van Wetenschappen Proceedings Series B Physical Sciences **18** (1916), 1520–1527.